Sample Project: Counting Kokanee

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Sample Project: Counting Kokanee

Orientation

Time:

Phase I: Experiment .... 1 class period
Phase II: Modelling .... 1 class period
Phase III: Verification of Model and Discussion.................. 1 class period

Instructional Strategies:

- Case Study
- Direct Instruction
- Group Work
- Simulation

Real-World Applications:

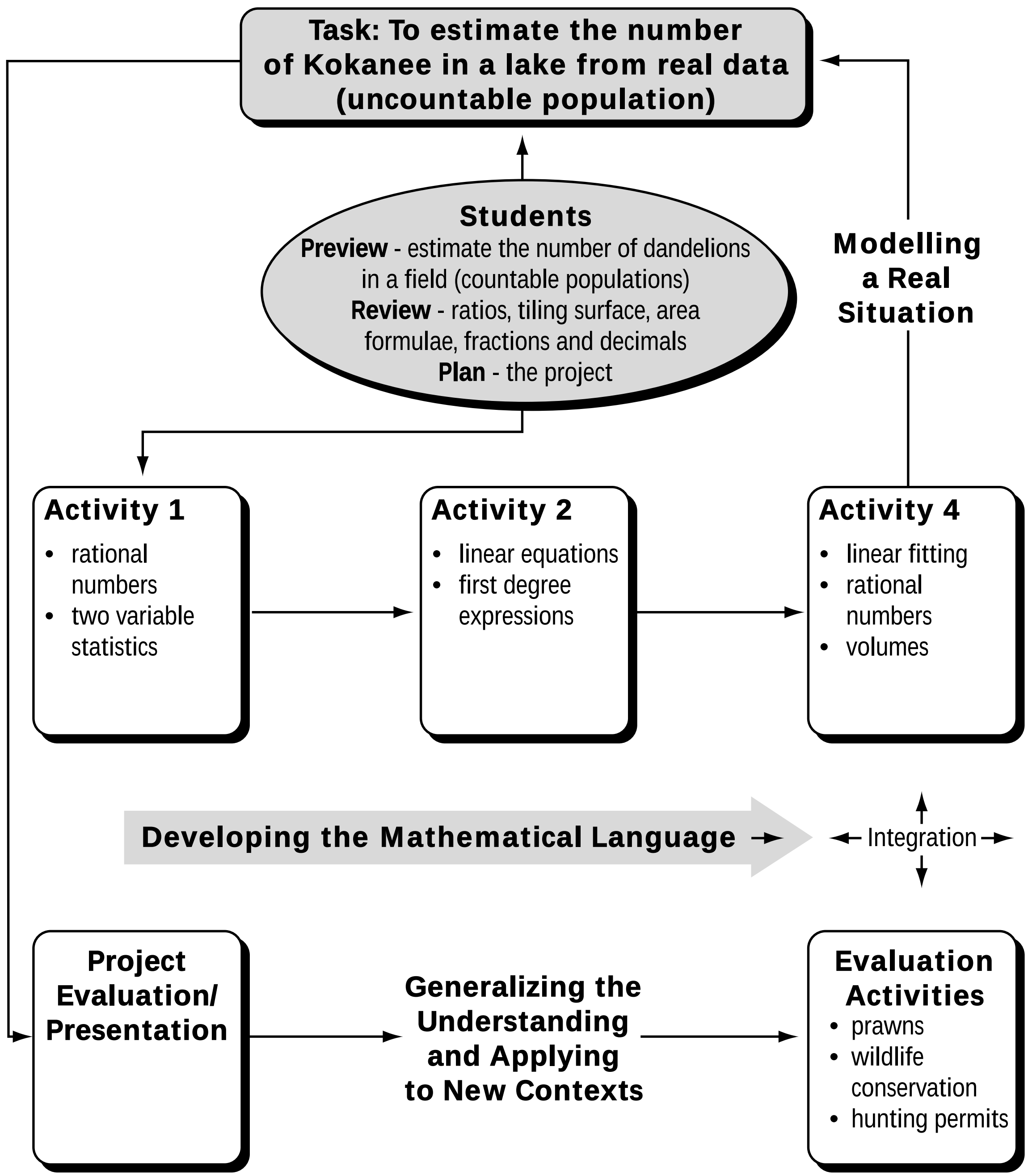
Some real-world applications of the skills learned in this project include:

- estimating the number of youth attending an outdoor rock concert
- estimating the number of salmon that have returned to spawn in a local river
- determining how many units of a trendy clothing item to stock
- doing quality control spot checks in a factory
- polling on a specific political issue
- spotting trends
- developing models to predict outcomes
- sampling populations
- demonstrating information in graph form

Project Overview:

Figure 1:Project Kokanee, is a schematic to help you conceptualize the Sample Project and begin the planning process with your students. Background information on mathematical modelling is provided at the end of this chapter. We recommend you work through that section before attempting to implement the Sample Project.

Figure 1: Project Kokanee —
'How to count everything without counting everything'



Project Synopsis

This two-hour sample project gives you and your students an opportunity to become familiar with the style of activities in *Mathematics 9: A Resource for Teachers*. The mathematics involved in the sample project is at the Grade 8 level and can be used as a review of ratio and proportion.

This project provides an opportunity for students to see how ratios and proportions are used in the real world.

In this project students construct a mathematical model that helps them estimate the number of Kokanee in a lake, and that can be extended to count other uncountable objects.

While working through the project, students should pay particular attention to establishing co-operative working relationships and to observing how a mathematical model is developed.

Background information on mathematical modelling is provided at the end of this project on Kokanee. Having students read this material, with your guidance, prior to beginning the sample project will help ensure that they get a consistent idea of how and why they might want to use a mathematical model. As they work through this sample project, you may find it helpful to discuss Figure 6: Modelling Flowchart now, and again at the end of the project.

In Phase I, students use the Capture-Mark-Recapture technique to estimate the size of a population. In a Math Lab, students represent fish by beads, and estimate the number of beads contained in a bag by applying the technique of Capture-Mark-Recapture. Students then assess the reliability of the technique.

In Phase II, students learn how to 'count the uncountable' by constructing a mathematical model. This model helps them estimate the number of Kokanee in a lake.

In Phase III, students verify and discuss the models that they developed.

The project introduces a 6-step process for modelling a real-world situation:
1. Identify a specific problem.
2. Identify variables that are inherent in the problem.
3. Determine possible relationships among the variables.
4. Represent the variables and their relationships numerically, graphically, and symbolically.
5. Verify the model in a constrained situation.
6. Validate the model by testing it in real-world situations.

The following questions can help guide the discussion for this project:
- What are some situations where it is important to estimate population size? For example, people at an outdoor concert, or bird populations.

- What are some smarter, faster and easier ways of determining the size of a population rather than by counting every single individual in the population?
- What are some indirect ways of determining the size of a population when you cannot count them one by one?

Summarize the discussion by explaining that we can:
- model the real world in the laboratory
- test and validate hypotheses
- apply our new knowledge to the real world.

Phase I: Experiment

Objective:

The objective of this phase is to have students determine situations where it is important to estimate populations and to have them learn a technique for doing this.

Materials:

For each pair of students:
- 1 large bag with approximately 400 to 500 white beads (this number is not disclosed to students)
- 1 small bag with between 15 and 40 red beads
- 1 jar or bowl

Procedure:

Have students identify situations where it is important to know or to estimate size of populations.

Have student brainstorm some indirect counting (estimation) procedures. Then have them describe the relevance of using any of these to determine the size of a population of fish.

Organize students into pairs and distribute the materials listed above.

Explain that students will be using a real-world technique commonly used by wildlife personnel. It consists of netting, marking, and releasing X number of fish. The marked fish are then given a period of time to disperse throughout the unmarked population. After a given period of time a new sample of size X is netted and the marked fish are counted.

Explain that the beads represent the population of fish in a lake and have each pair of students estimate the number of beads in the container. Each pair should record their initial estimate, and perhaps, their estimation techniques.

Each student pair should take a different size sample of between 15 and 40 beads from the large bag of beads and count them. These beads represent the fish that have been netted in the first sample. Each pair should record the sample size in their Student Activity table.

This sample is then replaced in the container by the same number of red beads. This process models the tagging of the fish.

Ask students to mix the beads thoroughly to represent the assumed natural mixing process of fish in the lake. Have students discuss some of the effects of inadequate mixing on future sampling.

Without looking in the bag, have each pair of students randomly remove the same number of beads as before.

In all likelihood, some of the beads in the second sample will be the same colour as those in the first sample and some will be red.

Count the number of red beads. These represent the tagged fish that have been netted in the second sample.

Ask pairs of students to compare and discuss the accuracy of their estimates for the first sample. Have each pair of students share their experimental results with the rest of the class.

Phase II: Modelling

Objective:

The objective of this phase is for students to learn and apply the basic principles of developing and testing models.

Materials:

For each pair of students:
- 1 large bag with approximately 400 - 500 white beads
- 1 small quantity of red beads
- 1 jar or bowl

Procedure:

What is the quantity you want to determine?
- the total number of normal colour beads in the bag that we will call N

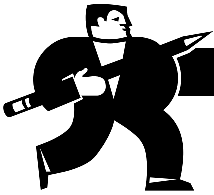
What are the quantities you know or have determined in this experiment?
- the size of the sample, that is the number of beads in the sample that we will call S
- the number of red beads in the second sample, that we will call D

Ask students to formulate the problem in their own words and to use the specific terms. This could be:

- Knowing the number of normal-coloured beads in a sample of S beads taken from a bag of beads, I must determine the number of beads contained in the bag.

Point out to students that a larger sample size generally has a larger number of different-coloured beads.

The students may collect all of the data from the other pairs of students and record the results in a table (Student Activity: Table of Mathematical Model). The students may also use their results to plot a graph showing the relationship of the number of different-coloured beads to the sample size (Student Activity: Graph of Mathematical Model).



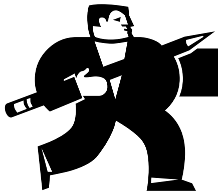
Student Activity: Table of Mathematical Model

Student Name: ____________________________ Date: ____________________

Table 1: Table of Mathematical Model

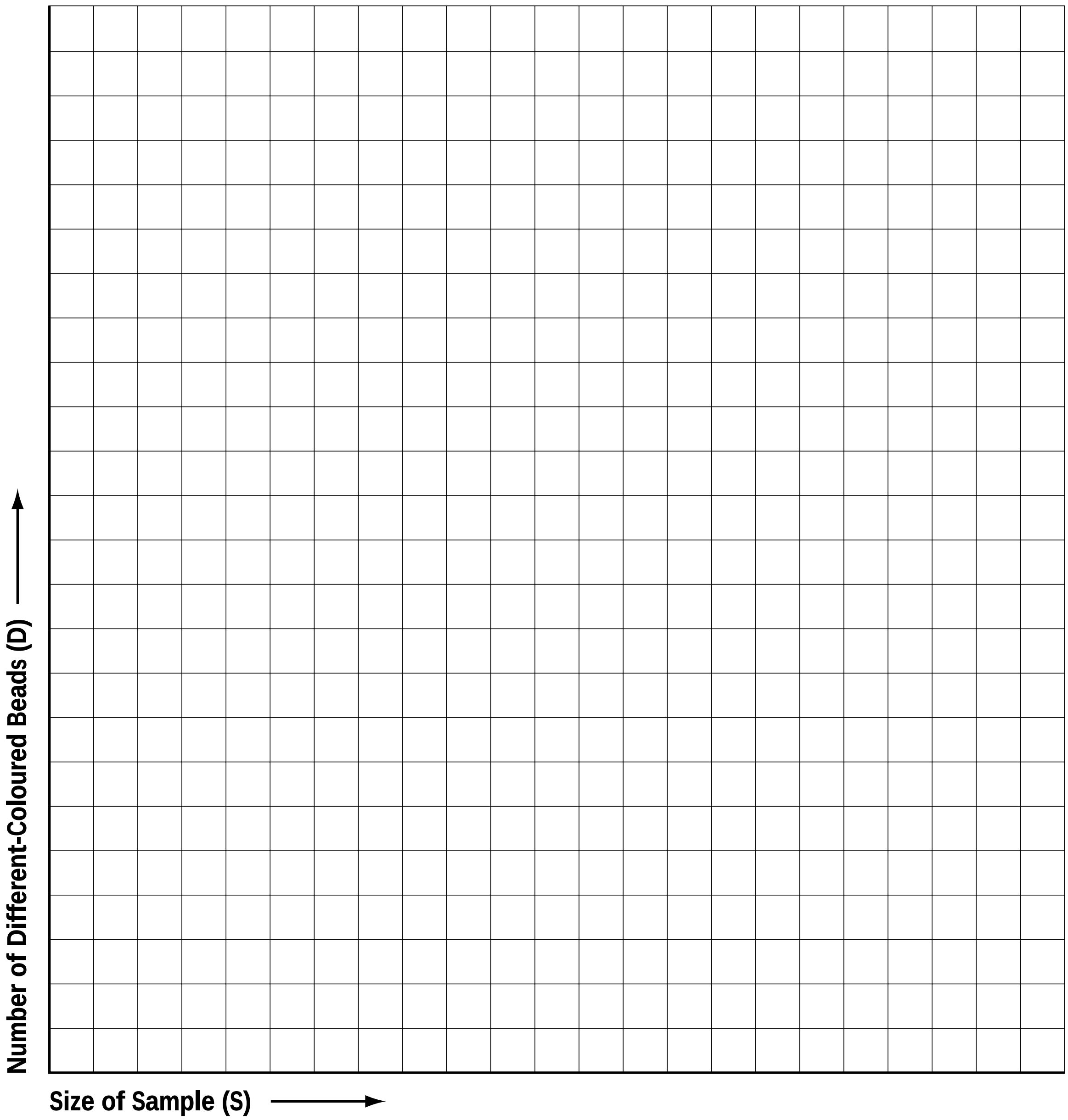
Student Pair	Size of Sample *S*	Number of Different-Coloured Beads *D*	Ratio $\frac{D}{S}$

Conclusions:



Student Activity: Graph of Mathematical Model

Student Name: ______________________ Date: ______________



Help students interpret the representations of the model from the table and the graph:

- referring to the table where the ratio calculated in the last column is constant, although some variations may be noticed, ask “How is the size of the sample related to the actual number of beads (population)?”.
- referring to the graph, ask “Are the data points situated on a straight line? What is the number of beads for a sample size of…)? What would the number be for a large sample?” Invite students to extrapolate the graph to a very large sample size. What does it mean? Do the students realize that a large sample could eventually be the total population?

Have students formulate the model with symbols from the following observation:

> the ratio of the number of different coloured beads to the size of the sample is the same irrespective of the size of the sample
>
> or
>
> D divided by S is the same number (a constant)
>
> or
>
> $\frac{D}{S} = k$ (k = a constant)

Assumption

Considering that the beads have been mixed thoroughly, we can assume that the proportion of the number of beads in the second sample is the same as the proportion of all the beads in the total population or

$$\frac{D}{S} = \frac{S}{N}$$

This assumption is based on the results obtained from either the table or from the graph. Whatever the number of beads, the same ratio will be obtained and this is true for the total number, that is, the total population.

Solve the Mathematical Model

Have students substitute the data into the model. Encourage them to first estimate and then use their calculators.

For example,
Size of sample: $S = 20$
Number of different-coloured beads: $D = 2$

Calculate N in the relationship $\frac{D}{S} = \frac{S}{N}$

$$\frac{2}{20} = 0.1 = \frac{20}{N}$$

$$N \times 0.1 = 20$$

$$N = \frac{20}{0.1}$$

$$N = 200$$

or

Size of sample: $S = 35$
Number of different-coloured beads: $D = 5$

Calculate N in the relationship $\frac{D}{S} = \frac{S}{N}$

$$\frac{5}{35} = 0.14 = \frac{35}{N}$$

$$N \times 0.14 = 35$$

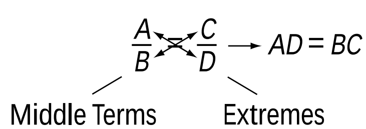
$$N = \frac{35}{0.14}$$

$$N = 250$$

Encourage students to determine a way of solving the equation systematically, for example, by using the property of proportionality illustrated below.

If $\frac{A}{B} = \frac{C}{D}$, then $A \times D = C \times B$ or the product of the 'extremes' equals the product of the middle terms:

Figure 2: Property of Proportionality



Ask students to practise the use of the property of proportionality with several results collected in the class. Then have students complete the table in Student Activity: Using and Testing the Model of Capture-Mark-Recapture.

In column one students record sample size. In column two they record the number of different-coloured beads. In column three they use the total population result obtained either in the previous table or graph they completed. In column four they record the total population results obtained from the mathematical model. Finally students estimate the deviation (E) between the actual experimental results and the formal result, and record it in column five.

They may consider expressing this deviation as a percent of the sample size in column six, that is, as a ratio of the deviation to the size of the sample.

Conduct a discussion on the results of Student Activity: Using and Testing the Model of Capture-Mark-Recapture. Have students comment on the size of the sample and its importance in the accuracy of the estimation.

Phase III: Verification of Model and Discussion

Objective:

The objective of this phase is for students to verify and discuss the model they developed.

Materials:

- Background on Mathematical Modelling section at the end of this project
- Figure 6: Modelling Flowchart at the end of this project

Procedure:

As you recall, modelling was presented as a 6-step process:

1. identify a specific problem
2. identify variables that are inherent in the problem
3. determine the relationship among the variables
4. represent the variables and their relationships
    - numerically
    - graphically
    - symbolically
5. verify the model in a constrained situation
6. validate the model by testing in real-world situations

The problem we identified was how to count a population of fish that could not be counted directly, other than by draining the lake! We assumed that we could estimate the population by sampling and then model this with a container of beads.

The variables we identified included:

- sample size *(S)*
- number of marked beads in the second sample *(D)*

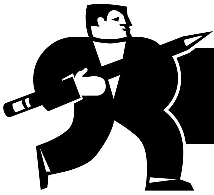
Through experimentation we represented the relationships numerically, graphically, and symbolically $\left(\frac{D}{S} = \frac{S}{N}\right)$

To verify the model, students should count the total number of beads in their containers and determine the accuracy of their experimental estimates.

As a class exercise, all the white beads could be placed in a large container and the procedure applied to the grand total. Does the model facilitate the determination of the total population?

Students should be able to suggest and describe situations where the model can be tested, and be able to design procedures that need to be followed in each case. Ask students to consider and discuss the following questions:

- In what ways does the model make common sense?
- In what ways does it make sense that the ratio must be the same irrespective of the size of the sample?
- What are some of the effects of insufficient mixing?
- What are some systematic ways that the model can be used to determine an unknown quantity?
- Does the model hold up when tested?
- What are some ways you know that the model is precise enough?
- What are some limitations of the model?
- What are some ways that the model can be improved?
- What are some ways that the model might be tested with real-world situations?



Student Activity: Using and Testing the Model of Capture-Mark-Recapture

Student Name: ____________________________ Date: ___________________

Table 2: Testing the Model of Capture-Mark-Recapture

Sample Size S	Number of Different-Coloured Beads D	Size of Population N_E (from Table or Graph)	Size of Population N_M (from Model)	Deviation $E = N_M - N_E$	% Deviation $\frac{E}{S}$ x 100

Conclusions:

Background Information on Mathematical Modelling

A mathematical model represents a system, that is, an assemblage of interdependent components that interact in various ways (e.g., solar system or the global economy).

Examining and analysing models can help students understand:
- how a particular system works
- what causes change in a system
- the sensitivity of the system to certain changes

Models can also help students predict:
- what changes will occur in the system
- when changes will occur in the system

There is a 6-step process for constructing a mathematical model to represent a real-world situation:
1. identify a specific problem
2. identify variables that are inherent in the problem
3. determine the relationship among the variables
4. represent variables and their relationships
   - numerically
   - graphically
   - symbolically
5. verify the model in a constrained situation
6. validate the model by testing it in real-world situations

There are many situations that can be modelled mathematically. For example, suppose an object is dropped from a great height. What are some ways that the fall can be described?
- How long will it take before the world runs out of oil?
- How far can a golfer hit a golf ball?
- What is the required dosage level and the interval of time between doses for a particular drug to be safe and yet effective?
- A chemical has spilled into a lake causing unsafe levels of pollution. Fresh water runs into the lake from a nearby stream, and water runs out of the lake at the same rate. How long will it be before the lake is cleaned to within 10% of the initial pollution level?
- Consider the guideline often given in driver education classes: allow one car length for every 15 km/h of speed under normal driving conditions. How good is this rule? What are some problems with this rule?

Procedure:

1. Identify a Specific Problem

We must be very specific in verbally formulating a problem in order to be successful in translating it into mathematics.

Given the problem of a falling object, there are still many questions we might ask:
- Does a heavier object fall faster than a lighter one?
- Does the object fall at a constant speed?
- If an object's speed changes during the course of the fall, how fast is the object moving when it strikes the earth?
- Does the object reach a constant maximum speed?
- Does the object fall halfway to the earth in half the time of the total fall?
- How long does it take for the object to reach the earth?

In order to successfully model this problem we must simplify the situation even further. For example:
What is the time of fall of an object of given mass dropped from a known height?
We need to be even more precise about the situation. Is it assumed that the object is falling from an aircraft, or simply released from rest? Are there forces that tend to slow the object as it falls? It is important to decide what the scope of the investigation is.
Therefore, let us suppose that:
- the object is released from different levels of a high-rise building
- the effects of the forces slowing the fall are disregarded.

2. Identify Variables That are Inherent in the Problem

It is not usually possible to capture all of the factors that influence the behaviour under investigation in a usable mathematical model. The task is to simplify the situation by reducing the number of factors, and to determine the relationships among the remaining variables. Again, we can reduce the complexity of the problem by assuming relatively simple relationships (linear or quadratic for example).

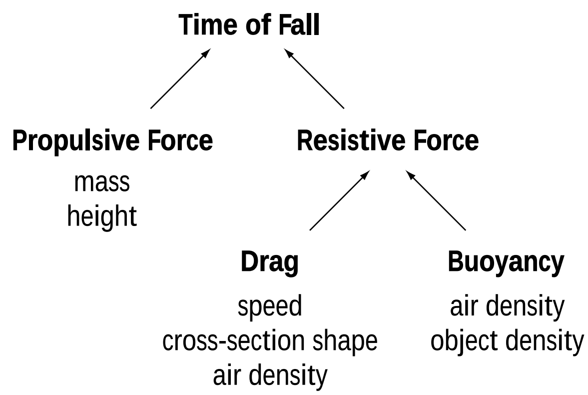
Classification of variables: dependent and independent

Variables are the factors that influence the behaviour identified in Step 1. Variables can be classified as dependent or independent. The variables the model seeks to explain are the dependent variables. The remaining ones are the independent variables. Certain variables can be omitted depending upon the questions asked.

Usually before determining relationships among variables, we must simplify the problem. If a problem is too complex, it may be preferable to study sub-models. In this case, one or more of the independent variables are studied separately. Eventually the sub-models are linked together under the assumptions of the main model.

In the 'falling body' problem, as we have described it above, time of fall is a dependent variable and the height of the object above ground is an independent variable. To simplify the problem we assume that the object is moving vertically only. What are the factors that affect the time of the fall or the object's position? Some forces are acting on the object (since it moves). Propulsion forces tend to move the object downward and resistive forces tend to diminish that motion. The resistive forces are caused by drag on the object as it falls and by the buoyancy of the air that supports the object and tends to hold it in space. The time an object takes to complete its fall depends on the propulsion and the resistive forces acting on it. To account for drag and buoyancy, we could create a sub-model that deals with the resistive forces.

Figure 3: Identification of Variables



3. Determine the Relationship Among the Variables

A flowchart of the relationships among the variables can be constructed and assumptions about the way each variable depends on the others can be made.

To further simplify the model you may choose to:
- ignore some of the independent variables. For example, the effect of one variable may be small compared to the others. You may choose to ignore a variable because you want to investigate the situation without the effect of this variable
- simplify relationships among the variables, for example, assuming the propulsion force is a constant multiple of the mass or that the drag equals some constant times the speed of the object. If subsequent tests of the model prove unsatisfactory, you can later refine the model by changing the assumptions.

4. Represent Variables and Their Relationships

- numerically
- graphically
- symbolically

All the sub-models are put together to obtain a numeric, graphic, and algebraic picture of the original situation. One often finds that the model is not complete or is so unwieldy that it cannot be interpreted. In those situations you must return to Step 2 and make additional simplifying assumptions. Sometimes it is even necessary to go back to Step 1 to redefine the problem. The ability to simplify or refine the model is an important part of modelling.

With respect to the ‘falling body’ problem, under the assumptions made in Step 2 and neglecting all the other variables for now, the algebraic representation is:

$t = kH/m$ where k is the constant of proportionality,
H is the height,
m is the mass of the object, and
t is time.

Note: the model is, of course, wrong since in the real world t is independent of the mass and depends on the square root of H.

5. Verify the Model in a Constrained Situation

Before using a model to reach real-world conclusions, it must be tested.

Several questions should be addressed before designing the tests and collecting data:
- Does the model answer the problem identified in Step 1, or did you stray from the key problem in designing the model?
- Does the model make sense?
- Can the data necessary to test and operate the model be gathered?
- Does the model hold up when tested?

In designing a test or an experiment for the model using actual data, you must include observations made over the same range of values for the independent variables as you would expect to encounter when actually using the model. The assumptions made in Step 2 may be reasonable over a restricted range of the independent variables yet very poor outside of those values.

In the ‘falling body’ problem, the model helps us determine the time of fall given a specific height. Furthermore, the model makes sense because as the height is increased, the time is longer (that is certainly what is expected). Moreover, it seems reasonable to think that heavier objects would fall faster than lighter objects. All other variables have been disregarded.

What are some ways in which the model can be tested in a constrained situation?

We could conduct some experiments by dropping a steel ball from different heights and measuring the time of the fall. This would take care of the ‘height of fall’ variable.

Then, we could drop balls of different masses from the same height in order to test the model with respect to the mass.

Finally, we could compare these experimental values against the values predicted by the model.

Graphs are very useful tools to assess models. Experimental results involving dropping a steel ball of a given mass from different heights are given in Table 3.

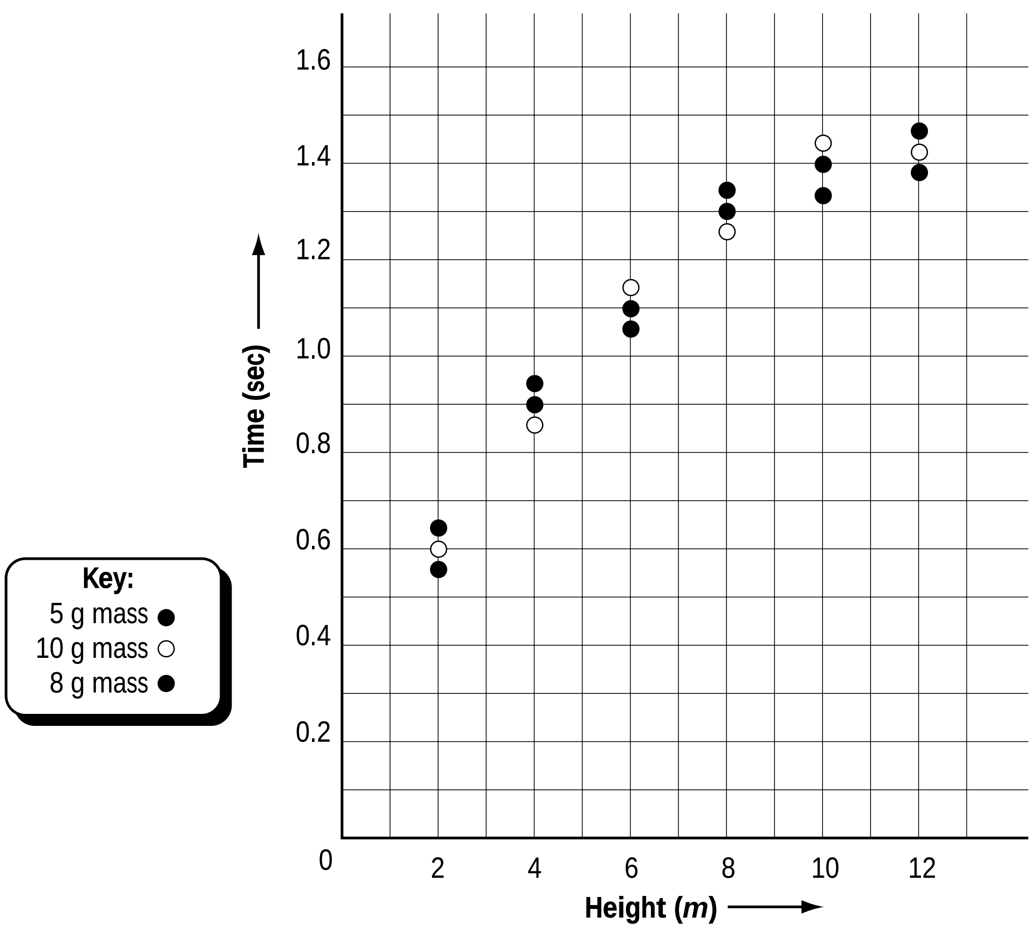
Table 3

Steel Balls Dropped From Different Heights

Height (*m*)	0	2	4	6	8	10	12
Time (sec)	0	0.6	0.9	1.1	1.3	1.4	1.5

Figures below show scatterplots of the data obtained with steel balls of different masses superimposed on the model (straight line).

Figure 4:Testing the Model: Steel Balls Dropped From Different Heights



The data points reveal that experimental results do match the straight line predicted by the model. However the experiments do not support the existence of a relationship between mass and time. The data shows that time is independent of mass.

From this we can conclude that there is something wrong with our assumptions.

Note: Be careful about conclusions drawn from an experiment. Broad generalizations cannot be extrapolated from particular evidence gathered about a model. A model does not become law simply because it is verified repeatedly in specific instances.

Assumptions need to be reviewed and modified in light of the experimental data. Experiments show that rather than being a linear relationship, the relationship between time and height tends to be more quadratic, and that mass has no influence on the time of fall.

A refined model could then be:

$$t^2 = kH$$

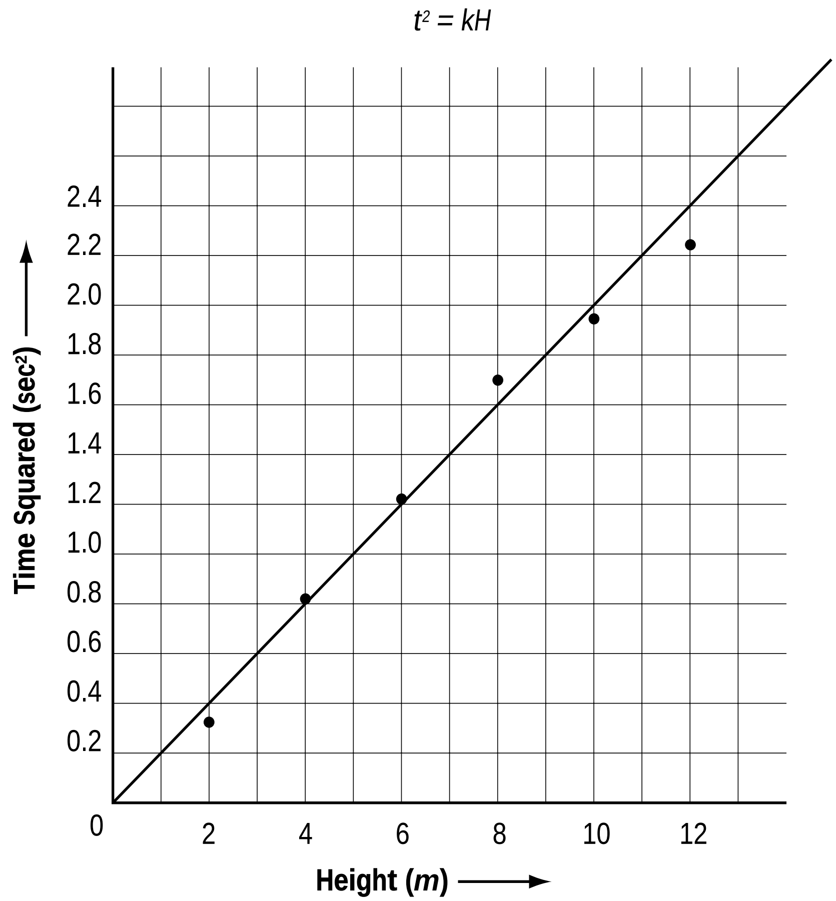
Representing the data in the following table and graphing the square of the time versus the height of fall would give the result shown in Figure 5: Testing the Model Under New Assumptions.

Table 4

Testing the Model Under New Assumptions

Height (*m*)	0	2	4	6	8	10	12
Time (*t*) Squared	0	0.36	0.81	1.21	1.69	1.96	2.25

Figure 5: Testing the Model Under New Assumptions



Refining the model: Part 2 (Extension)
A careful examination of the graph suggests that the time values are becoming too small for larger values of the height. If this is indeed a pattern that holds for larger values of *H* obtained in supplementary experiments, the model will need to be reconsidered in order to take this information into account. Perhaps the object is experiencing some force that tends to slow its downward motion, such as atmospheric drag. If this is the case, then larger values of *H* must be considered.

6. Validate the Model by Testing in Real-World Situations

One of the reasons for developing models is to help us predict events in the real world.

In order to see if our model is adequate, students could use the model to predict the time of fall of objects when dropped from various heights (e.g., tennis ball, basketball, baseball). Having calculated a value, they could record data using the real items, graph the results, and comment on the validity of the model.

Iterative Nature of Model Construction

Model construction is an iterative (repeating) process. The flowchart below shows the steps to be taken.

Figure 6: Modelling Flowchart

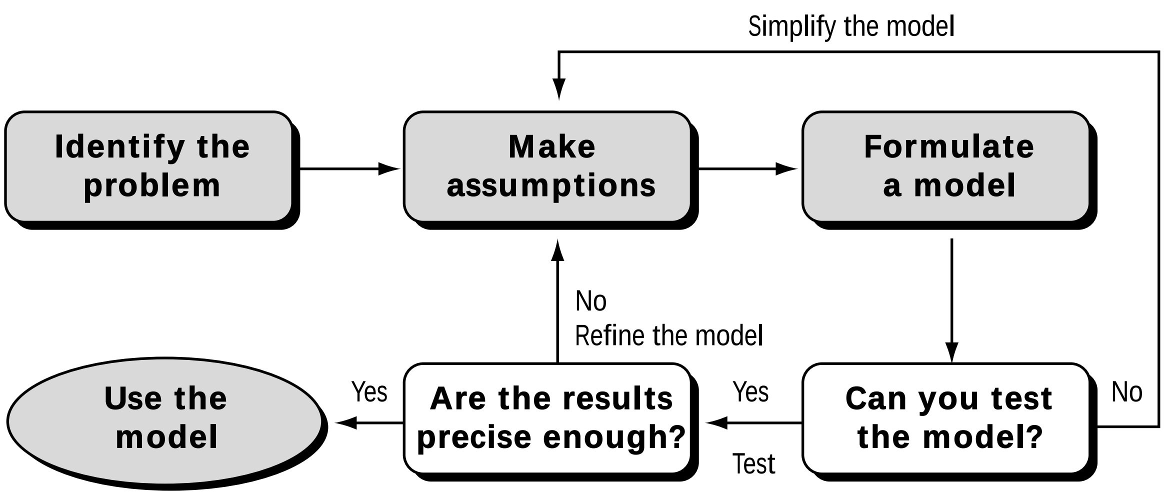


Table 5

Simplifying or Refining the Model as Required

Model Simplification	Model Refinement
• restrict problem identification	• expand the problem
• neglect variables	• consider additional variables
• combine the effects of several variables	• consider each variable in detail
• set some variables to be constant	• allow variation in the variables
• assume simple (linear) relationships	• consider nonlinear relationships
• incorporate more assumptions	• reduce the number of assumptions