Project A: Predicting Satellite Collisions

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Orientation

Time:

Phase I: Space Junk
Introduction and Review .... 1 class period
Research .... 2 class periods
Planning .... 1 class period
Phase II: Speakers
Gears .... 1 class period
Speakers .... 2 class periods
Beehives .... 2 class periods
Phase III: Iso- and Geosynchronous Orbits
Achilles and the Turtle .... 1 class period
Planets .... 2 class periods
Andromeda and the Liquid-Mirror Telescope .... 1 to 2 class periods
Big Bang .... 2 class periods
Autopsy of a Collision .... 1 + class periods

Instructional Strategies:

- Direct Instruction
- Case Study
- Group Work
- Issues Inquiry

Real-World Applications:

Some real-world applications of the skills learned in this project include:
- reading and understanding a scientific article
- projecting job trends, house prices, and business sales
- describing growth and decay of populations
- projecting the increased value of collectables like hockey cards, comic books, Beanie Babies, and Barbie dolls
- making good consumer decisions on the purchase of mountain bikes and stereo speakers
- using logarithmic scales like the Richter Scale
- using precision and accuracy
- optimizing the use of space

- packing camping gear more efficiently
- using exponents and appreciating their efficiency

Project Overview:

Figure 1 on the following page is a schematic to help you conceptualize Project A, Predicting Satellite Collisions.

Project Synopsis

In this project students learn more about the risk of collision between satellites, and between satellites and fragments of satellites or meteorites. The risk of collision has increased because of the dramatic increase of satellites placed into specific orbits. This issue is becoming an important problem that people should be prepared to deal with in the very near future.

Through reading a scientific article and doing research, students are introduced in a rigorous way to an real-world socio-technological issue. They are expected to read and gather information about the issue and to develop a critically-informed opinion about it. By doing this, they realize the importance of having a sufficiently developed mathematical and scientific background. In particular, they become aware that estimating the probability of a collision requires the extensive use of exponents and a working understanding of formulae involving rational numbers.

After reviewing the concepts related to the different representations of rational numbers, the concept development of the project begins with the exploration of the limitations of the set of rational numbers. By dealing with real-world situations, students are introduced to phenomena where continuity is involved. By modelling situations with formulae involving powers and by applying the laws of exponents, students expand their mathematical skills and abilities, improve their pattern representation skills, and deepen their understanding of an actual socio-technological issue.

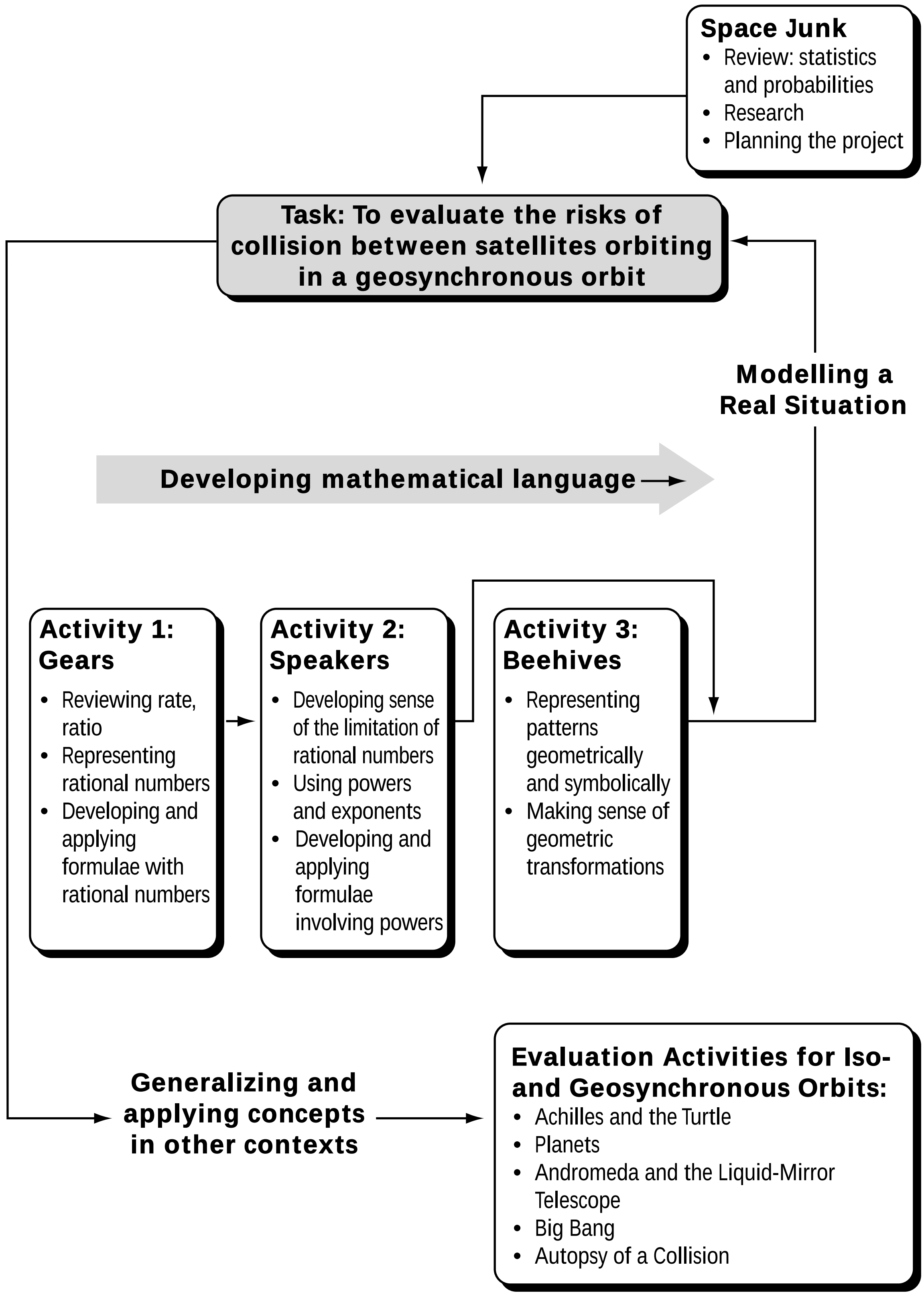
Phase Synopsis

The project is structured in the following way:

In Phase I: Space Junk, students read a scientific article in order to describe and analyse a situation. They then propose a plan of action based on their acquired knowledge.

In Phase II: Speakers, students model real situations with rational expressions and develop new skills and abilities needed to improve their model. By doing three activities they explore the limitations of rational numbers in order to deal with situations where continuity is involved. The activity sequence is designed so that students progressively realize the limitations of models using rational numbers.

Figure 1: Project A: Predicting Satellite Collisions



In Phase III: Iso- and Geosynchronous Orbits, students increase their awareness of the necessity of modelling situations with formulae involving powers to represent patterns. They refine their skills in manipulating algebraic expressions containing powers with integral components and rational bases as a simplification tool. In this phase, students focus on understanding the algebraic expressions rather than on developing manipulation skills. Students use calculators extensively to help them visualize, and are introduced to symbolic software in order to simplify algebraic expressions. In this phase students also propose a refined model by applying their new knowledge and identify further improvements that are needed to reflect more closely the actual data.

Phase I: Space Junk

Activity 1: Introduction and Review
Activity 2: Research
Activity 3: Planning

In this project students are asked to collect relevant data from as many sources as possible and to prepare a report on the cluttering of outer space. They are asked to present their findings in graphical form, and to discuss the possible consequences of this environmental problem.

There are two major objectives in Phase I. The first is to review Grade 8 concepts of number, pattern, statistics and probability. The second is to serve as a case study that helps most students develop new concepts and deepen their understanding of the problem. To assist them in their planning in Phase I, teachers may wish to consider the following options:

Option A:
Students are asked to prepare a report that includes the following:
- an introductory statement about the issue and an outline of the actual concerns;
- graphical presentations of the data that may include: population of satellites and debris versus size, population of satellites versus altitude and orbits, increase in the number of satellites over time;
- a discussion, illustrated by graphs, of the natural space cleaning processes; solar effects and return to earth; and
- a general conclusion.

Option B:
Students may also include:
- an outline of the mathematical and scientific concepts that would need to be developed in order to improve the model
- a graph illustrating a breakup and a collision
- a discussion of surveillance systems used today and their ability to predict collisions
- a flowchart or diagram showing different aspects of the problem

Space Junk is divided into three activities:

In Activity 1: Introduction and Review, the teacher introduces the project, indicates the importance of the issue, explains to students what is expected from them, and makes suggestions for the type of data that students need to collect.

The teacher suggests several sources of information for students.

This is also an opportunity for the teacher to informally assess whether students have the prerequisites to do this activity. Prerequisites include:

- knowledge and use of scientific notation
- knowledge of the concepts related to probability and statistical data
- skill at representing a set of data in graphical form in order to recognize patterns
- skill in using a graphing and/or a scientific calculator
- knowledge of force and energy (from the Science curriculum)

In Activity 2: Research, the teacher uses examples to demonstrate the advantages and disadvantages of using graphs. In particular, students are exposed to the limitations of extrapolating from a graph.

The teacher provides examples to emphasize the importance of using scientific notation and of modelling phenomena using formulae with rational numbers.

Students are expected to prepare their report from the information collected. The teacher helps students to:

- decide upon the kind of outcome they are seeking
- find the best way to plot their data on a graph in order to identify patterns; and
- determine the relative influence of various factors on the graphs

In Activity 3: Planning, students present their report to the class. The teacher establishes a sequential list of mathematical and scientific concepts that are helpful in understanding this particular issue as well as other similar issues.

At the conclusion of this first phase, the teacher presents the project plan and indicates the additional concepts that will be developed in Phase II: Speakers, and Phase III: Iso- and Geosynchronous Orbits.

Activity 1: Introduction and Review

Objective:

This activity has three main objectives. First, students will have a reasonable understanding of the key aspects of the issue. In particular, it is expected that they will be able to identify the factors involved in the increase in the amount of space debris and number of satellites. The second objective is the review of number concepts and operations acquired in Grade 8. The third objective is for students to gain confidence in their ability to read and comprehend scientific articles.

Materials:

- reprint of the article by Nicholas L. Johnson. "Monitoring and Controlling Debris in Space", *Scientific American* (August 1998): 62-67.
- Student Activity: Space Junk Introduction and Review

Procedure:

Introduce the activity and ask students to read the Scientific American article. Assist them in gaining understanding by guiding discussion about key aspects.

Use the data presented in the article to:
- verify that students are able to use their calculators correctly, and
- review scientific notation, ratios and proportions, properties of the circle, projects of distance, mass, time, and velocity.

Have students answer the questions suggested in Student Activity: Space Junk Introduction and Review and complete the tables from the analysis of the 3-D graph in Figure 1.

Students may either produce a cardboard model of the 3-D graph or reproduce the graph using computer graphics.

Background Article:

> On June 3, 1996, the upper stage of a Pegasus rocket launched in 1994 broke up creating the largest debris cloud on record - more than 700 objects large enough to be tracked, in orbits from 250 to 2500 kilometers in altitude. This single event instantly doubled the official collision hazard to the Hubble Space Telescope that orbits just 25 km below. Further radar observations revealed an estimated 300 000 pieces of debris larger than four millimeters, big enough to damage most spacecraft. During the second Hubble servicing mission in February 1997, the space shuttle Discovery had to maneuver away from a piece of Pegasus debris that was projected to come within 1.5 kilometers. The astronauts also noticed pits on Hubble equipment and a hole in one of its antennas, that had apparently been caused by a collision with space debris before 1993. (Nicholas L. Johnson, "Monitoring and Controlling Debris in Space", *Scientific American* [August 1998], 62-67).

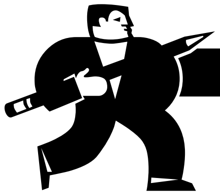
Additional Background Information:

Satellites revolve around the earth in specific orbits. The most overcrowded of these altitudes are 850 km, 1000 km, 1500 km, the semi-synchronous altitude of 20 000 km, and the geosynchronous altitude of 36 000 km in the plane of the equator. The lifetime of a satellite is quite short and no systematic and efficient way of disposing of the 'dead' satellites has yet been implemented.

Furthermore, satellites decay naturally, producing more and more debris. This debris collides and creates more debris. Even microscopic debris like paint flakes of 0.1 mm or less poses a risk of a dramatic collision. For example, the solar panels of the Hubble space telescope are punctured by paint flakes and solid-rocket motor slag. When this happens, the efficiency of the panels is greatly reduced. All of these factors contribute to increased risk of a serious collision between an active satellite and debris.

Currently, the probability that an average size satellite (10 m^2) will be destroyed is 1/50 000 per year and is increasing exponentially (in 1995, the probability was 1/100 000). This probability depends upon the density of satellites at a particular altitude. Actually, one satellite orbiting between 800 km and 1000 km is expected to be destroyed within the next 10 years.

The risk of collision is taken very seriously by those people responsible for the construction of the International Space Station. A piece of microscopic debris could be fatal to an astronaut working outside the station.

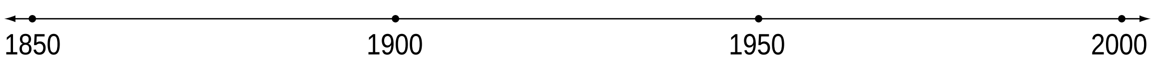


Student Activity: Space Junk Introduction and Review

Student Name: ____________________ Date: ______________

After reading the article by Nicholas L. Johnson. “Monitoring and Controlling Debris in Space”, *Scientific American* (August 1998): 62-67. Complete the following time line by indicating the number of identifiable objects in space at particular points in time.

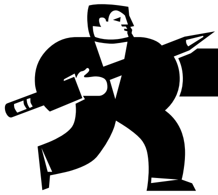
Figure 2: Exploration of Space



Approximately 9000 objects (10 cm or more in length) are orbiting the earth. About 550 of these are active satellites. Fifty-four of these active satellites are equipped with electronuclear reactors. Electronuclear reactors contain a nuclear carburant, uranium 235 for Russian satellites, or plutonium 238 for North American satellites. This nuclear carburant has a mass of 1.45 tons.

Using the above information from the article, estimate the following:

- the ratio of active satellites to the total number of space objects
- the ratio of nuclear satellites to the total number of satellites
- the average mass of an orbiting object in kilograms
- the average mass of an active satellite in kilograms
- the average mass of nuclear carburant per nuclear satellite in kilograms
- the closest orbit to earth in kilometres
- the altitude of the geosynchronous orbit in kilometres



Student Activity: Space Junk Introduction and Review

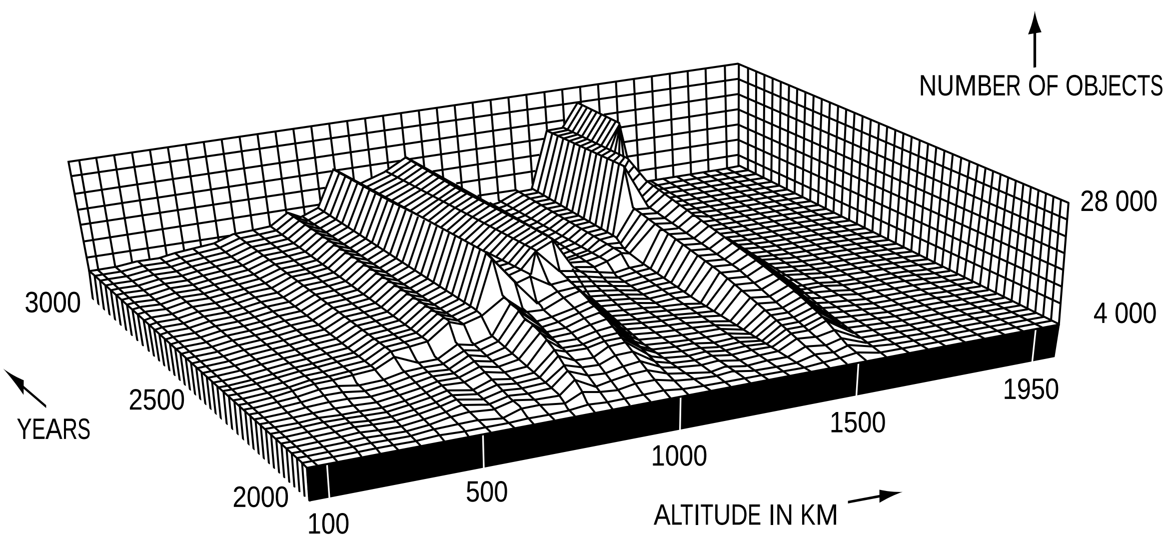
- the velocity of a satellite orbiting at 1000 km

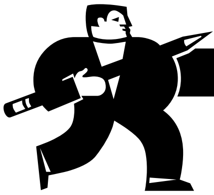
- the velocity of a satellite orbiting at 2000 km

- the velocity of a satellite orbiting at 36 000 km

Using information from Figure 3: Number of Objects in Space, complete Tables 1 to 4 and the comment areas for Tables 3 and 4.

Figure 3: Number of Objects in Space





Student Activity: Space Junk Introduction and Review

Table 1

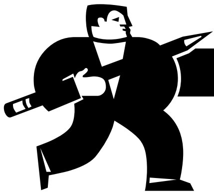
Space Junk in 1998

Altitude (km)	Number of Objects	Mass of Objects (tons)
100		
500		
750		
1000		
1250		
1500		
1750		
2000		
35 785		

Table 2

Space Junk Projections for 2050

Altitude (km)	Number of Objects	Mass of Objects (tons)
100		
500		
750		
1000		
1250		
1500		
1750		
2000		
35 785		



Student Activity: Space Junk Introduction and Review

Table 3

Increased Density of Space Junk 1997 to 2050

Altitude (km)	% Increase in Number of Objects	% Increase in the Number of Tons
100		
500		
750		
1000		
1250		
1500		
1750		
2000		
35 785		

Comments:

Table 4

Projections of Space Junk for the Year 2100

Altitude (km)	Number of Objects	Mass of Objects (tons)
100		
500		
750		
1000		
1250		
1500		
1750		
2000		
35 785		

Comments:

Activity 2: Research

Objective:

The objective of this activity is to help students be aware of the need to increase their knowledge and skills so that they can correctly interpret graphical representations of real-world data.

Materials:

- reprint of the article by Nicholas L. Johnson. "Monitoring and Controlling Debris in Space", *Scientific American* (August 1998): 62-67.
- access to Internet
- graphing calculator and computer with graphics software
- Student Activity: Planning the Research

Procedure: Part 1

Help students in their second reading of the *Scientific American* article and have them interpret the graphs.

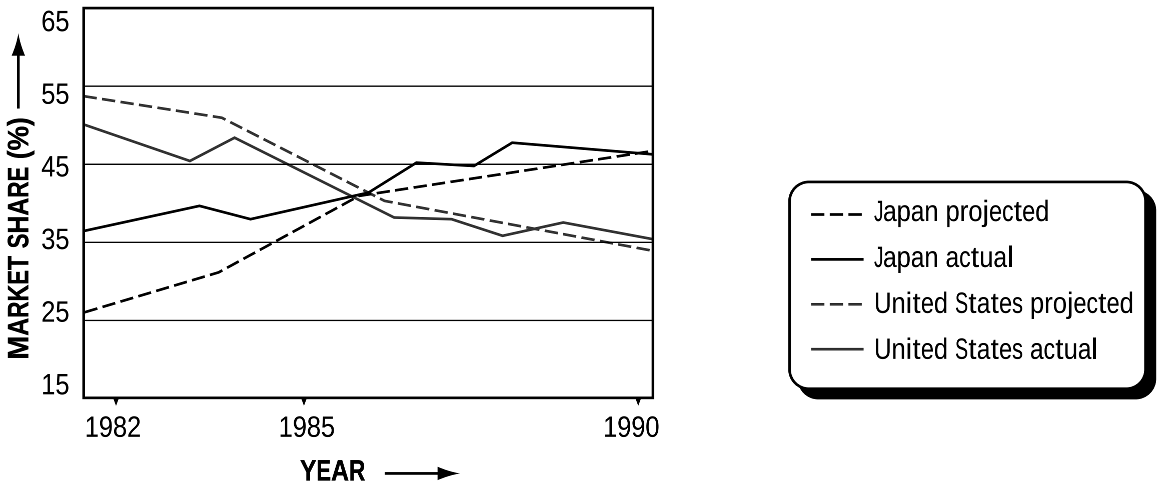
Tell students that extrapolating from graphs requires both a good knowledge of the various factors involved in a situation and on a good understanding of the relationships between and among the variables of a problem situation.

Introduce and comment upon the following situation:

> In 1989, a team of researchers at M.I.T. (Massachusetts Institute of Technology) prepared a report on Industrial Productivity. They examined the reasons American dominance of the high-technology industry was experiencing a serious competitive decline. Problems identified ranged from poor product quality to excessively long product cycles. The researchers presented the following graphical representation of the situation and projected the future of the industry to the year 2000.

Present an overhead of the graph, Figure 4: Market Share of Global Semiconductor Industry for Japan and the United States, 1982-1989. Have students use appropriate vocabulary to comment on the shape of the two lines: Japan Projected and United States Projected.

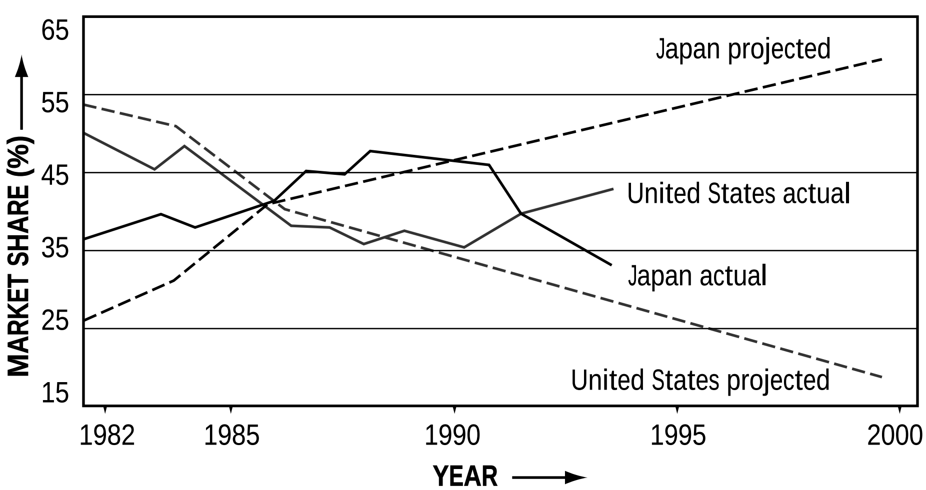
Figure 4: Market Share of Global Semiconductor Industry for Japan and the United States, 1982-1989



Adapted from: R.K. Lester, *The Productive Edge*: *How US Industries are Pointing the Way to a New Era of Economic Growth*, New York: W.W. Norton and Company, 1998.

Present an overhead of the graph, (Figure 5: Market Share of Global Semiconductor Industry for Japan and the United States) representing the actual situation in 1998 and superimpose this on an overhead of the previous graph (Figure 4: Market Share of Global Semiconductor Industry for Japan and the United States., 1982-1989).

Figure 5: Market Share of Global Semiconductor Industry for Japan and the United States



Adapted from: R.K. Lester, *The Productive Edge*: *How US Industries are Pointing the Way to a New Era of Economic Growth*, New York: W.W. Norton and Company, 1998.

Have students use general terms to comment upon the discrepancy between the projected situation in Figure 4 and the actual situation in Figure 5.

Ask students to identify some of the factors that were not considered in the projection but have since been revealed to be of crucial importance. The turning point for this is in 1991.

Procedure: Part 2

Use the previous example to stress that it is necessary to continue to identify all of the possible factors affecting the evolution of a situation and to determine how these factors are related. Use the flowchart of Modelling Situations as a guide to conducting this discussion.

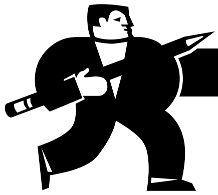
Have students identify the factors involved in the present situation and have them discuss the possibility of relationships among orbits, mass, energy and velocity. Demonstrate the revolution of an object about a fixed point in space, attached with different lengths of rope to relate the length to the velocity.

Ask students to recall and discuss the different types of satellites, their number, and their different orbits.

Inform students about the construction of the International Space Station and the risks of an accident. The Discovery Channel website provides additional information about the International Space Station.

Have students complete Student Activity: Planning the Research.

Ask students to discuss their projections from Figure 6: Estimate of Earth's Artificial Satellites in 1998 to Figure 7: Projection of Earth's Artificial Satellites and Debris for Year 2025. Then compare these with the results shown on the 3-D graph of Activity 1, Space Junk - Introduction and Review.



Student Activity: Planning the Research

Student Name: ____________________________ Date: ___________________

Complete the following:

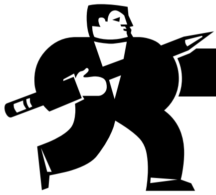
What are some reasons it is important to be able to monitor the amount of debris in Space?

What are some reasons it is important to be able to control the amount of debris in Space?

If you were in charge of a team having to propose solutions to the space junk problem in outer space, list and briefly describe some specific kinds of information you would need.

What are some of the ways that you could obtain this information?

What are some of the ways you would like to see this information presented? Why?



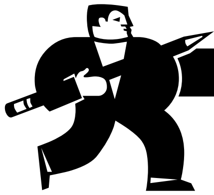
Student Activity: Planning the Research

Complete Table 5: Information.

Table 5

Information

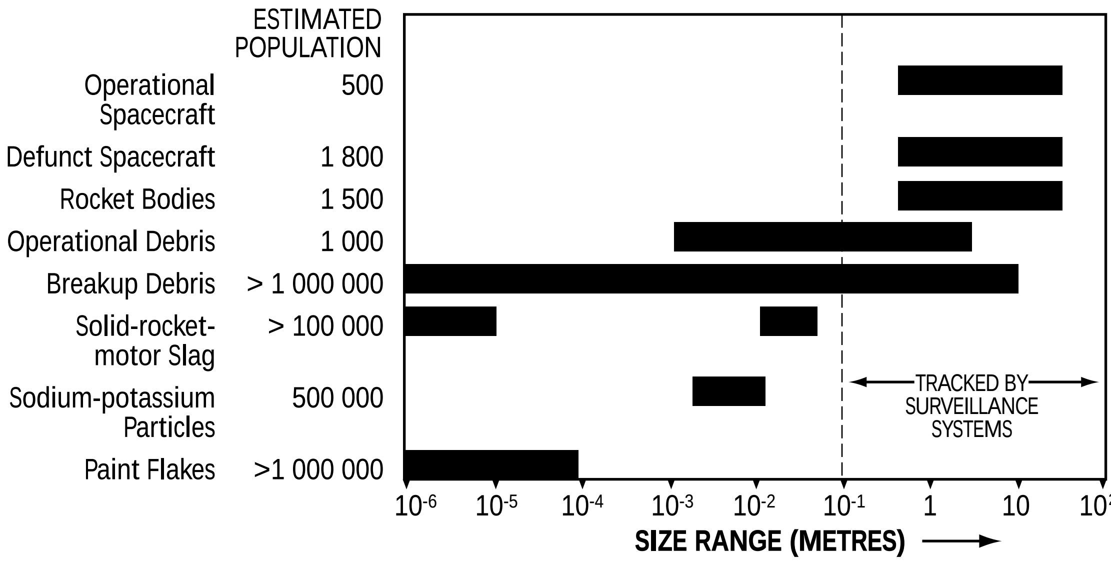
Kinds of Information to be Collected	Methods for Collecting the Information	How the Information will be Used

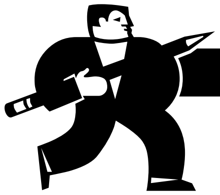


Student Activity: Planning the Research

The following bar graph (Figure 6: Estimate of Earth's Artificial Satellites in 1998) represents the 1998 estimated population of space craft and debris according to size.

Figure 6: Estimate of Earth's Artificial Satellites in 1998

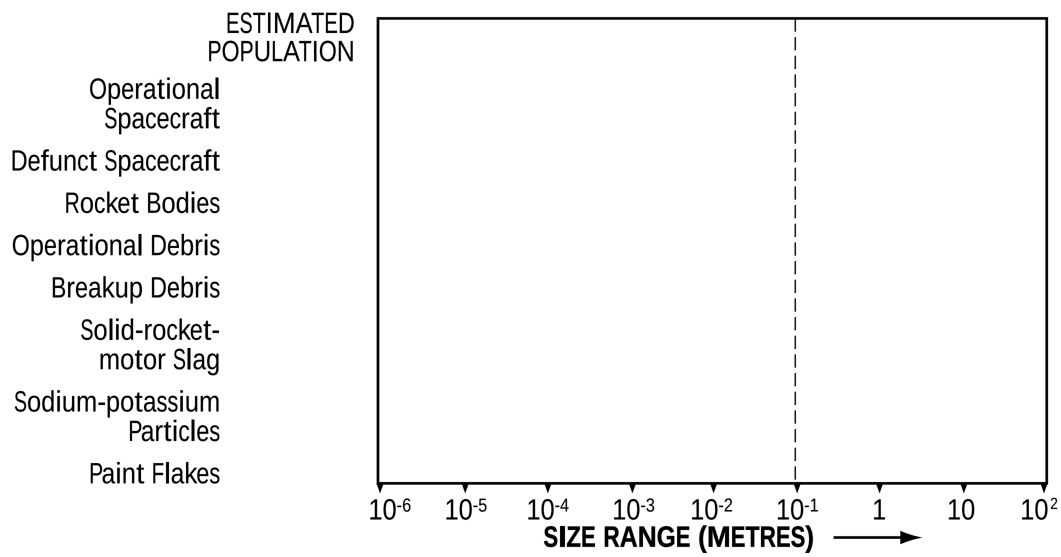




Student Activity: Planning the Research

Create a similar bar graph representing the situation in the year 2025. List and describe the factors you considered in creating your graph.

Figure 7: Projection of Earth's Artificial Satellites and Debris for Year 2025



Provide your reasons for each population estimate.

Operational Spacecraft

Defunct Spacecraft

Rocket Bodies

Operational Debris

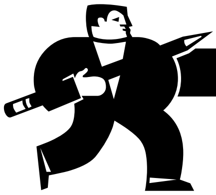
Breakup Debris

Solid-rocket-motor Slag

Sodium-potassium Particles

Paint Flakes

Comments:



Student Activity: Planning the Research

List and briefly describe all of the factors that you think are important to consider.

In Table 6: Risk Factors, regroup the factors according to whether they increase or decrease the risk of collisions.

Table 6

Risk Factors

Factors that Increase the Risks	Factors that Decrease the Risks

Activity 3: Planning

Objective:

Under the teacher's guidance, students prepare a flowchart for their Project Action Plan by completing the Self Evaluation Checklist and the Questions to Guide Your Project Planning.

Materials:

- Student Activity: Planning the Research
- graphing calculator
- computer with graphics software

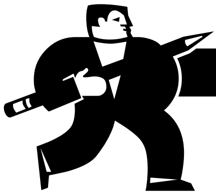
Procedure:

Ask students to describe the issue verbally. Provide guidance to them so that they come to realize that there are two aspects to the problem:
- an increase in the number of satellites and consequently an increase in the number of tons of debris, and
- a cleaning process whereby there is a natural return to earth of some debris due to the action of the sun.

Ask students to propose a method to determine the probability of a collision between debris and an operating satellite, spacecraft or Space Station. The method proposed should involve the concept of density of an orbit.

Ask students to develop a Project Action Plan Flowchart. In order to complete their Project Action Plan Flowchart, students are asked to identify the mathematical and scientific concepts they need to explore, and to describe their expectations regarding the application of these concepts in the present context. The creation of the flowchart should follow a brainstorming session about all of the factors involved and how these factors are related.

The teacher presents the plan and establishes its parallel with the flowcharts suggested by the students.



Student Activity: Planning the Research

Student Name: ____________________________ Date: ___________________

1. Self Evaluation Checklist

Rate yourself on each of the following. Circle the number that corresponds with how well you are satisfied with your performance in each area. Put a check mark in front of areas in which you would like more help.

Key: 1-Not satisfied, 2-Need some help, ... 5-Very satisfied

Help?	Areas	Rating
❑	using negative exponents	1 2 3 4 5
❑	representing a number in scientific notation	1 2 3 4 5
❑	expressing a ratio in different ways	1 2 3 4 5
❑	representing and applying percents in fractional or decimal form	1 2 3 4 5
	performing operations with rational numbers in	
❑	fractional form	1 2 3 4 5
❑	decimal form	1 2 3 4 5
❑	using rates, ratios, proportions to solve problems	1 2 3 4 5
❑	using patterns to analyse and solve a problem	1 2 3 4 5
	representing patterns in the form of	
❑	graphs	1 2 3 4 5
❑	equations	1 2 3 4 5

2. Questions to Guide Your Project Planning

Clearly identify and describe the problem in writing. Be specific.

Phase II: Speakers

Phase Synopsis

The intent of this phase is to review and to reinforce the students' abilities regarding the two representations of the rational numbers - fractions and decimals - and to explore the limitations of the rational numbers. Students realize that, in some cases, reality is modelled with numbers that are not expressible in fractional form. This is the case with the spectrum of musical frequencies and with the limit polygons inscribed in a circle. In each case, students are exposed to a continuous situation that can be approached by successive approximations with a calculator or a computer. Students realize that numerical approximations can be made as close as they wish and that these are accurate enough for most applications.

Through a series of five activities, students practise their skills in order to model real situations in a variety of familiar contexts. To do this, they take advantage of their ability to recognize and represent relations between the different factors they have identified as playing a significant role in the situation.

In Activity 1: Gears, students are provided with an opportunity for review and introduction of modelling with formulae involving rational numbers. Students determine the gear ratios of a mountain bike and create a cardboard model of the gear mechanisms of a clock (Big Ben).

In Activity 2: Speakers, students learn about modelling with rational numbers and are introduced to the limitations of the set of rational numbers. Students compare different speakers by understanding frequency response curves provided by stereo speaker manufacturers. Using this information, students establish the mathematical models that will permit them to determine the frequency of a musical note from a discrete spectrum to a continuous spectrum. By re-scaling the intensity of a sound into decibels, they also explore the powers of 10.

Activity 3: Beehives allows students to evaluate the concepts covered in this phase. Students construct cardboard models of a beehive to help them evaluate their understanding of rational numbers.

Activity 1: Gears

Objective:

There are two major objectives in Activity 1. The first is a review of rate, ratio and proportion concepts as well as a review and reinforcement of pattern recognition skills. The second objective is an increased understanding of rational numbers, their representation in decimal and fraction forms, and their usefulness in real-world situations.

Materials:

- bicycle (10 , 15 or 18 speed)
- hooks
- rope
- chalk
- measuring tape
- cardboard
- scissors
- tape
- pins
- geometry kit
- Student Activity: Gears

Activity 1.1: Spur Gears

In this activity, students establish the relationship between the gear ratios of a bicycle and the distance corresponding to a complete revolution of the pedals. Determining this information helps them optimize their gear changing when operating their bicycles.

Procedure:

Have students measure the radius of the back wheel and the diameter (or radius) of each of the spur gears. A carpenter's compass may be used for this.

If the weather permits and if space is available, have students measure the distance travelled on the ground during 10 complete revolutions of the pedals for each gear combination.

If the weather does not permit doing the activity outdoors, have students suspend the bicycle and measure the number of turns of the back wheel corresponding to a complete turn (360^o) of the pedals for each of the 18 gear combinations. Make sure that the students hold the back wheel so that there is no free rotation during the measurements. Have students make a chalk mark on the tire so that they always start at the same place. Have students pedal backwards to position the pedal vertically before starting each new measurement.

To increase precision, students may measure the angles and distances of additional revolutions of the pedals and then divide by the number of revolutions. That is, three complete revolutions of the pedal gives five and one-third turns of the back wheel. This means that one revolution corresponds to 16/9 turns.

When they finish this task, have them complete the tables in Student Activity: Gears.

Ask students to develop a rule about the relationship between the chain ring diameters and the number of teeth (spurs) on each chain ring. To do this, they will need to do the following:

- examine all of the chain rings on the rear cog set;
- measure the diameter of each chain ring;
- measure the distance between the teeth on each chain ring;
- count the number of teeth on each chain ring; and
- determine the rule.

Establish a rule for the relationship between the diameter ratios and the ratios of the number of teeth.

Have students describe the relationship between the different ratios and the effort needed. Ask questions such as “How many more times do you need to pedal to go the same distance with different gear ratios?” In order to calculate the number of complete revolutions they need to pedal for a given distance, for example 150 meters, they may utilize Table 12: Gear Turns and Distances.

Have students develop a rule giving the number of complete pedal revolutions needed to go a given distance for a given gear ratio.

Activity 1.2: Angular Velocity

In this activity students determine the relationship between the gear radii and the angular velocity of a following gear in order to understand the mechanisms of a clock or a watch.

Have students construct cardboard models of spur gears like the ones illustrated in Figure 8 : Angular Velocity. Make sure that the dimensions (width and spacing) of the teeth are all the same. Each group of students constructs sets of gears of different radii.

Help students determine the number of complete turns that the gear rotates per second, for example, 1/3 turn per second. This is the angular velocity. Have them express angular velocity in terms of p per second, for example, 1p per sec, 1/4 p per sec, and revolutions per second or per minute, that is, r.p.s. or r.p.m.

Have students attach two gears using a piece of cardboard acting as an arm (see Figure 9: Rotating Gears).

Holding the larger gear in place, students rotate the arm with some angular velocity. If they rotate the arm too fast, the smaller gear will slip. Ask students to determine the angular velocity of the small gear. They estimate the answer, for example twice as fast, and determine the precise answer using a stopwatch.

In order to determine how fast the small gear is rotating (that is its angular velocity), ask students to complete Table 13: Velocities and then to develop a formula by trial and error. Encourage groups of students to collect results from other groups.

The answer is:
v_a is the angular velocity of the arm
v_s is the angular velocity of the small gear
R_L is the radius of the large gear
R_s is the radius of the small gear

the angular velocities are related by:

$v_S / v_a = (R_S + R_L)/ R_S$

Figure 8: Angular Velocity

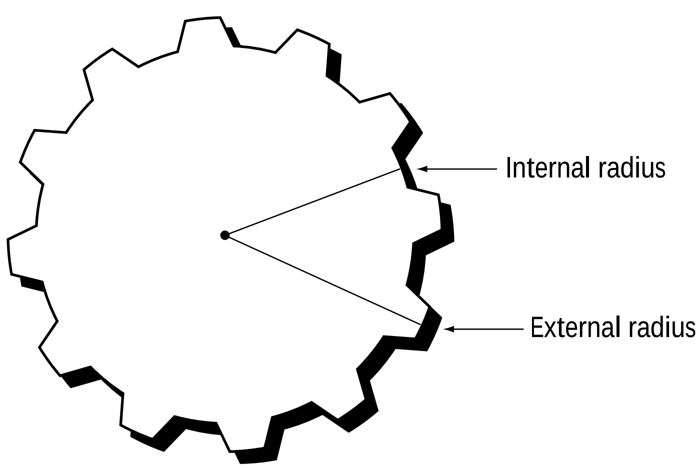
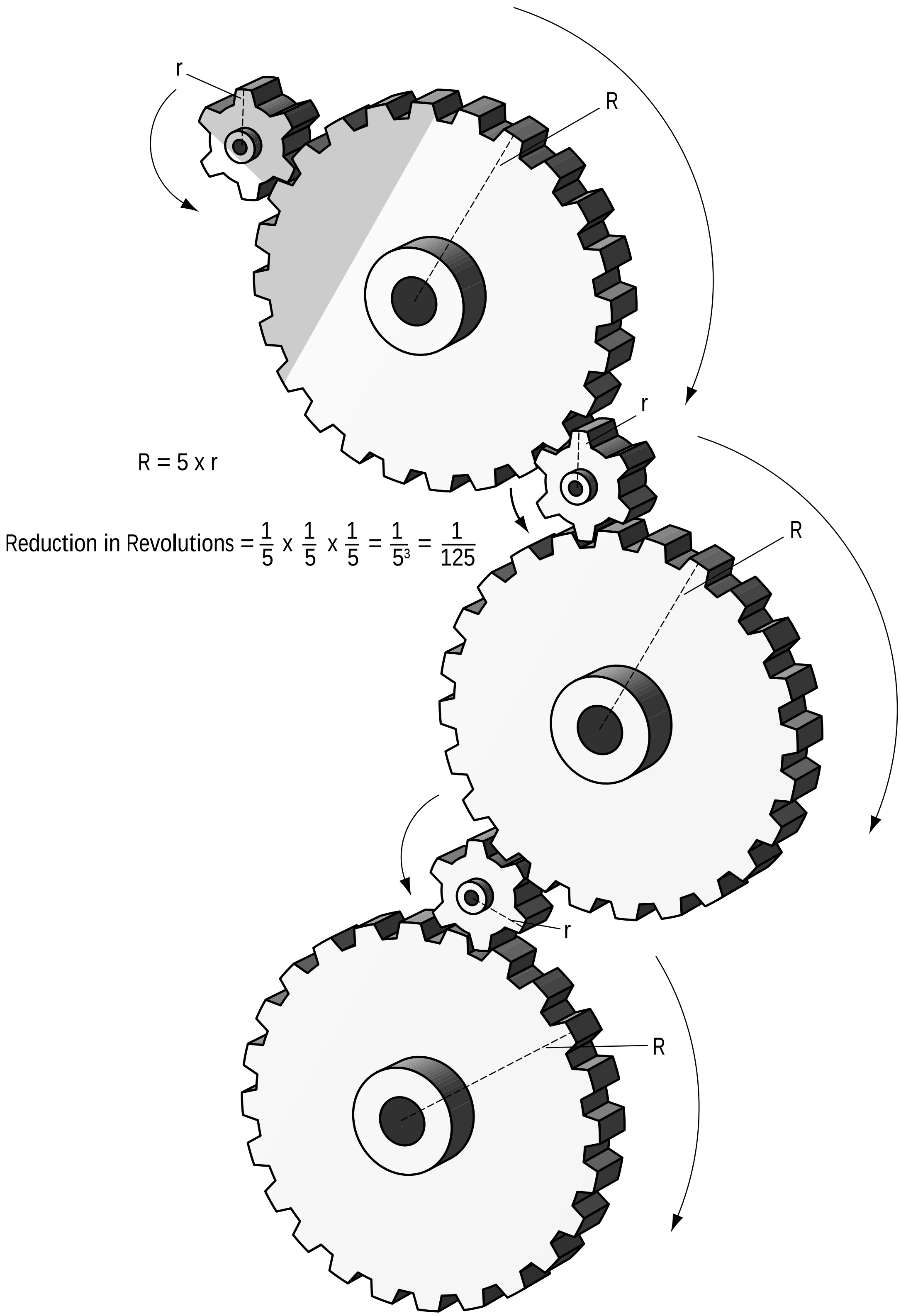


Figure 9: Rotating Gears



Challenge 1: Drive Shaft

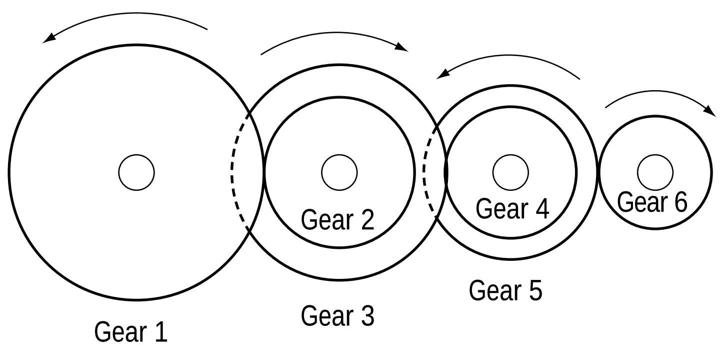
Two parallel shafts are centered 30 cm apart. If the drive shaft makes 80 r.p.m., find the diameters of wheels that work by rolling contact so that the driving shaft makes 240 r.p.m.

Apply this information about wheel diameters to determine the radii of a sequence of 5 gears needed to reduce the angular velocity by a factor of 100. For example, an electric motor has a r.p.m. of 1200 and one needs to rotate an arm with an angular velocity of 12 revolutions per minute or 1/5 of a turn per second.

Challenge 2: Big Ben

Old clocks and watches were built by assembling a series of spur gears to form a certain train. This activity demonstrates how this works. Students may choose to carry out research on Big Ben in order to appreciate the precision of the mechanism of this famous clock.

Figure 10: Spur Gears



Six spur gears are mounted upon parallel shafts. They engage as shown in Figure 10: Spur Gears. The number of teeth is as follows:

Gear #1: 84
Gear #2: 72
Gear #3: 48
Gear #4: 52
Gear #5: 36
Gear #6: 24

Find the angular velocity ratios v_2/v_1, v_3/v_1, v_4/v_1, v_5/v_1 and v_6/v_1

The last ratio, v_6/v_1 , is called the value of the train. It is equal to the product of the number of teeth in all the drivers, divided by the product of the number of teeth in all the followers.

What are the results if some of the intermediate shafts have one gear instead of two, and if one gear acts as both a driver and a follower?

Project Link:

Suppose that the orbit of the earth around the Sun is a circle whose radius is the average distance of the aphelion and the perihelion (see Tables 7, 8, 9, and 10: Physical Data for Planets).

Determine the average distance of the aphelion and the perihelion.

Determine the Earth's angular velocity and express it in different units.

Determine the velocity of the earth in orbit in km/sec.

In astronomy, physical data are often expressed in 'Earth Units', that is, by dividing the physical data of a planet by the corresponding data of the earth.

Express the following in Earth Units:

- the radii of the planets
- the periods of revolution in earth years
- the angular velocity in Earth Units.

Table 7

Physical Data for the Planets #1

Body	Radius (km)	Mass (kg)	Average Density (g/cc)	Semi-Major Axis of Revolution (km)	Aphelion (km)	Perihelion (km)
Sun	695950	1.991 E30	1.410			
Mercury	2433	3.181 E23	5.431	5.795 E7	6.986 E7	4.604 E7
Venus	6053	4.883 E24	5.226	1.081 E8	1.0885 E8	1.9737 E8
Earth	6371	5.979 E24	5.519	1.4957 E8	1.5207 E8	1.4707 E8
Moon	1738	7.354 E22	3.342	3.844 E5	4.055 E5	3.633 E5
Mars	3380	6.418 E23	3.907	2.2784 E8	2.4912 E8	2.066 E8
Jupiter	69758	1.901 E27	1.337	7.7814 E8	8.1580 E8	7.405 E8
Saturn	58219	5.684 E26	0.688	1.4270 E9	1.5045 E9	1.3495 E9
Uranus	23470	8.682 E25	1.603	2.8703 E9	3.002 E9	2.738 E9
Neptune	22716	1.027 E26	2.272	4.4999 E9	4.537 E9	4.463 E9
Pluto	5700	1.08 E24	1.65	5.909 E9	7.375 E9	4.443 E9
Icarus	0.53	5.0 E12	8.1	1.613 E8	2.949 E8	2.77 E7

Note: Icarus is asteroid # 1566

The exponents for large numbers are expressed as in the following example:
1.991 E30
is equivalent to
1.991×10^{30}

Table 8

Physical Data For the Planets #2

Body	Sidereal Period of Revolution (sec)	Equatorial Rotation Period (sec)	Average Solar Day (sec)	Rotational Velocity at Equator (km/sec)	Altitude of an Isosynchronous Satellite (km)
Sun		2.125 E6		0.6085	24054600
Mercury	7.602 E6	5.068 E6	1.5204 E7	0.003	237298
Venus	1.941 E7	R 2.107 E7	R 1.01 E7	0.0018	1534693
Earth	3.156 E7	8.616 E4	8.64 E4	0.4651	35767
Moon	2.361 E6	2.361 E6	2.551 E6	0.0046	86673
Mars	5.936 E7	8.864 E4	8.878 E4	0.2400	17049
Jupiter	3.743 E8	3.543 E4	3.543 E4	12.6568	87688
Saturn	9.296 E8	3.684 E4	3.641 E4	10.296	48797
Uranus	2.651 E9	R 3.881 E4	R 3.881 E4	3.8930	36370
Neptune	5.200 E9	5.665 E4	5.665 E4	2.5194	59506
Pluto	7.837 E9	5.523 E5	5.523 E5	0.0648	76510
Icarus	3.5338 E7	9.72 E4	9.72 E4	0.0000453	264

Note: The letter ‘R’ indicates a retrograde or opposite motion to that of the Sun/Earth system

Table 9

Physical Data for the Planets #3

Body	Gravity (Mean) (cm/sec^2)	Gravity (Polar) (cm/sec^2)	Gravity (Equator) (cm/sec^2)	Escape Velocity (km/sec)	Orbital Velocity Perihelion (km/sec)	Orbital Velocity Aphelion (km/sec)
Sun	27372.0	27370.0	27369.0	617.23		
Mercury	357.8	357.8	357.8	4.1725	58.92	38.82
Venus	887.4	887.4	887.4	10.365	35.256	34.780
Earth	980.7	985.1	975.0	11.179	30.272	29.278
Moon	162.0	162.2	162.0	2.3735	1.075	0.964
Mars	374.0	376.5	355.7	5.0282	26.490	21.964
Jupiter	2601.0	2850.0	2260.0	60.238	13.700	12.435
Saturn	1117.0	1291.0	863.0	36.056	10.177	9.129
Uranus	1049.0	1156.0	937.0	22.194	7.116	6.490
Neptune	1325.0	1325.0	1297.0	24.536	5.472	5.383
Pluto	221.0	221.0	221.0	5.023	6.102	3.676
Icarus	0.120	0.357	0.069	0.00112	93.458	8.794

Table 10

Isosynchronous Satellite Data

Body	Altitude (km)	Orbital Velocity (km/sec)	Escape Velocity from (km/sec)	Solar Escape from (km/sec)
Sun	24055600	73.19	103.5	
Mercury	237301	0.297	0.420	67.64
Venus	1534720	0.459	0.650	49.53
Earth	35775	3.073	4.346	42.33
Moon*	86667	0.235	0.333	1.48
Mars	17032	1.447	2.046	34.18
Jupiter	89304	28.21	39.89	43.96
Saturn	50964	18.62	26.33	29.65
Uranus	36938	9.780	13.83	16.84
Neptune	59504	9.119	12.82	15.01
Pluto	76470	0.935	1.322	6.828
Icarus	264	?	?	?

Moon*: Escape from Earth

Student Activity: Gears

Student Name: ____________________________ Date: ___________________

Complete the following tables

Table 11

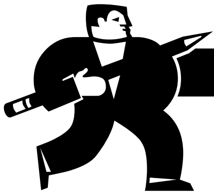
Dimensions

	Diameter (cm)	Number of Spurs
Front #1		
Front #2		
Front #3		
Back #1		
Back #2		
Back #3		
Back #4		
Back # 5		

Table 12

Gear Turns and Distances

Front	Back	Diameter Ratio	Spur Ratio	# Turns Back Wheel	Total Angle	Distance	Comments



Student Activity: Gears

Table 13

Velocities

Group ID	r_L	r_S	$(r_L + r_S)\ r_S$	v_a	estimated v_S	v_S

Formula :

Questions:

What is the angular velocity if the two gears have the same radius?

Are you surprised by this result? Draw a figure or roll a dime around a loonie to convince yourself.

What is the angular velocity if the radius of the small gear is half that of the larger gear?

What is the angular velocity if the radius of the small gear is five times smaller than that of the larger gear?

What is the angular velocity if the radius of the small gear is ten times smaller than that of the larger gear?

What is the rule?

What are some ways that you can modify the formula when you know the ratio of the two radii of the gears?

Activity 2: Speakers

Objective:

Students analyse the specifications of various speakers based upon the Frequency Response Curves information provided in manufacturers' brochures and booklets. Students perform a cost-benefit analysis based on their budget and their own musical requirements. At the end of this activity, students should be able to compare the features of different speakers.

Students apply concepts related to rational numbers and powers. They then apply formulae involving rational numbers and powers, and use mathematical models of physical situations.

Materials:

- manufacturers' information booklets on various types of stereo speakers
- metal strings of different lengths and compositions
- keyboard, piano, or other string instrument
- Student Activity: Speakers
- Student Activity: Range of Emitted and Perceived Sounds

Activity 2.1: Frequency Response Curve

When purchasing a set of stereo speakers, it is important to be able to identify the most important factors. In this activity students analyse the specifications and the frequency response curves relating the intensity of sound to the frequencies of speakers.

Procedure:

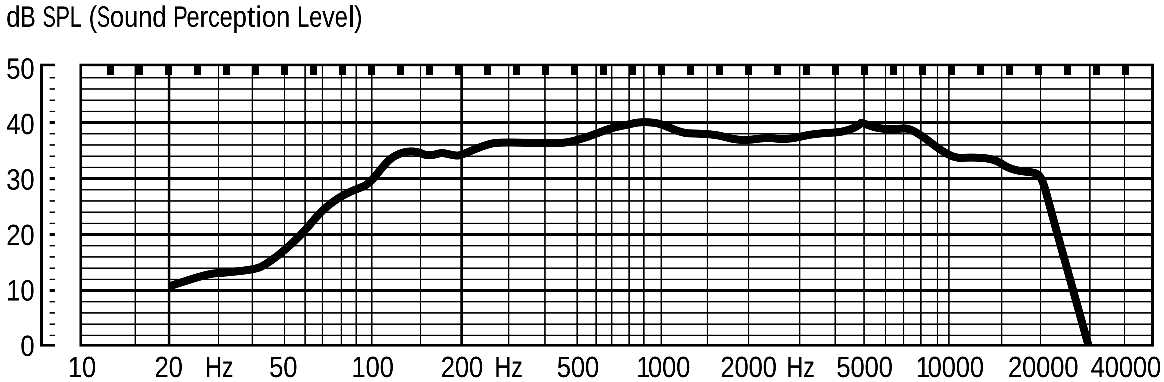
As preparation for this activity, ask students to go to music stores to obtain information about the various types of speakers on the market. The students should collect pamphlets that include information about the Frequency Response Curves (FRC) of the speakers, maximum power, power handling, frequency response range, and price.

You may invite a music store representative to make a brief presentation to the class about speakers and FRCs.

Select a typical FRC (Figure 11: Typical Frequency Response Curve) and project this onto an overhead screen. Ask students to comment on the features of the FRC graph by answering the following questions:

- What do the symbols dB and Hz mean?
- Why is the frequency scale not a linear one?
- Why is there a plateau between two frequencies?
- Why does the intensity drop rapidly after a certain frequency?
- Why is the intensity zero for some frequencies?
- Why do all the scales start at the same 20 Hz frequency?

Figure 11: Typical Frequency Response Curve



Have students complete a list of the features for all of the speakers they analysed (see Table 15: Speaker Specifications).

Tell students that they are going to learn more about these so that they can be better consumers when they choose to buy a set of speakers. To become better consumers, they need a better working knowledge of rational numbers and powers.

Present students with the two charts of Student Activity: Range of Sound and have them comment upon the charts.

Have students brainstorm different ways to produce musical sounds. Sounds are produced by vibrating strings, air forced through a pipe, vibrating membranes, and vocally. Give examples of musical instruments. Remind students that sound is characterized by a frequency. Help students define the frequency as the inverse of the period, that is, the time needed for a complete vibration. Invite students to estimate the period of a vibrating string and, then, to estimate the frequency of the vibrating string.

Present students with the frequency units (reciprocal sec, Hz, number of vibrations per second). Referring to the sound range charts in Student Activity: Range of Emitted and Perceived Sounds, ask questions such as:

- What is the range of periods of the sound emitted by humans?
- Are the sounds of cats of a lower or higher frequency than those made by dogs?
- What advantage would there be to a killer whale having a hearing range greater than its sound emission range? How many times greater is that range?

Activity 2.2: Vibrating Strings: The Violin

In this activity, students establish the relationship between the frequency of a vibrating string and the tuning of musical instruments. The frequency is derived from observations and experiments. Students also identify three characteristics of sound - intensity, pitch, and tone.

Procedure:

Before providing students with the formula for the frequency of a vibrating string (violin, piano,), ask them to identify the variables involved. They should mention the length, tension, diameter and the material of the string.

Tell students that rather than considering the nature and diameter of the string, it is preferable to characterize the string by its linear density. Linear density is the weight of a 1 metre length of string. For example, if 1 metre of string weighs 50 grams or 0.05 kilograms, the linear density would be 0.05 kilograms/metre.

Have students experiment with strings of different lengths and linear densities in order to determine how different variables affect the frequency. It is easier for students to estimate the period instead of the frequency and then to take the reciprocal.

Have students complete the qualitative table below (Table 14: The Effects of Different Variables on Frequency). Explain that the up and down arrows are qualitative measures and simply indicate increase or decrease in a variable, and a double arrow indicates a greater increase than a single arrow. For example, in line 1, we increase length and period, and we keep tension and linear density constant. This results in a frequency decrease.

Table 14

The Effects of Different Variables on Frequency

Length (*L*)	Tension (*T*)	Linear Density (*D*)	Period (*P*)	Frequency (*F*)
↑	constant	constant	↑	↓
constant	↑↑	constant	↓	↑
constant		constant		
constant	constant	↑		
↓	constant	constant		
↓↓	constant	constant		
constant	↓↓	constant		
constant	constant	↓↓		
constant	↓	↑↑		

Give students the formula for frequency (n)

$$n = k\ \frac{1}{2L}\sqrt{\frac{T}{D}}$$

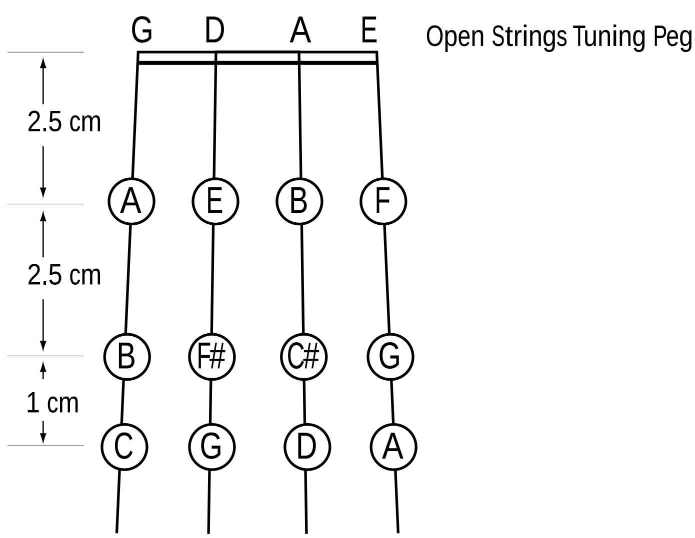
where T is the tension in Newton (kgm/s^2, D is the linear density in kg/m and L is the length in m, k is the number of Hz partially used (usually k = 1).

Have students verify that the units are correct by replacing each variable by its own unit. The frequency should then be expressed in Hz or reciprocal seconds (s^{-1}).

Have students apply this formula to a violin like the one illustrated in Figure 12: Open Tuning.

The length of the four strings between the tuning peg and the bridge is 26 cm.

Figure 12: Open Tuning



Ask questions such as:

The lowest note that can be obtained is the C_1 note (approximately 16 Hz) and the highest is the C_9 (approximately 16 700 Hz). The only instrument having this range is the organ.

- Can the human ear perceive those two notes?
- Can a dog perceive those two notes?
- Can a killer whale perceive those two notes??

How is the frequency changed when:

- pressing one string with the first, the second and the third finger?
- the tension of a string is decreased?
- the tension of a string is increased?
- the tension in a string decreases from 25 kg/mm^2 to 16 kg/mm^2? (answer: the second frequency will be 4/5 of the first)

Activity 2.3: Rational and Irrational Numbers

In this activity, students develop a formula to evaluate musical frequencies from a standard frequency used in orchestras to tune all the instruments. They are exposed to the limitations of the rational numbers.

Procedure:

For historical reasons, the musical sounds have been grouped into notes and scales. Each scale is divided into 12 notes (8 + 4):

C or B#
C# or D b
D
D# or E b
E or F b
F or E#
F# or G b
G
G# or A b
A
A# or B b
B or C b

Musicians use 10 successive scales (octaves) that is, a total of 121 notes covering the entire range of human audible sound from the extreme low to the extreme high. The lowest note C_1 has a frequency of approximately 16 Hz and the highest note C_9 has a frequency of approximately 16 700 Hz.

In two consecutive octaves, one note has a frequency that is exactly double the corresponding note of the previous octave. For example, in the second octave the frequency of the C_2 note would be approximately 16 x 2 = 32 Hz, in the third, C_3 has a frequency of 16 x 4 = 64 Hz, then 128, 256, 512, 1 024, 2 048, 4 096, 8 192, 16 384 Hz.

Ask students to use Student Activity: Tuning an Instrument in order to develop a formula that gives them a tool to determine the frequency of a given note from the frequency of the previous note of the same octave. (The answer is multiply by $\sqrt[12]{2}$). Guide students by giving them the following hint:

- suppose that the frequency of the next note is obtained by multiplying the frequency N by some number (call it x). Repeat the same process until you reach the same note in the next octave

$$N\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x = Nx^{12}$$

because the frequency of the same note in the next octave is twice as large as the starting frequency.

$2N = Nx^{12}$
or $x^{12} = 2$
or $x = 2^{1/12} = \sqrt[12]{2}$

This number is not a rational number because any root of a number that is not a perfect square is not rational. Rational numbers may always be expressed as fractions.

Tell students that non-rational numbers can only be approximated. For example, students may use their calculators and perform the square root () function twenty times on the number 2 and then perform the SQUARE function twenty times but they will never get the number 2 exactly. In fact, after 11 operations, the rounding of the result has an effect on the answer.

Most musicians tune their instruments from a very specific note, the *A* 440. Ask students to devise a strategy in order to evaluate the frequency of the middle *C* on a piano. (Answer: 3 notes higher, that is, $2^{1/12}$ x $2^{1/12}$ x $2^{1/12}$ = $2^{1/4}$ = 1.1892071 x 440 = 523.25 Hz)

Students should use their calculators to estimate the value of $\sqrt[12]{2}$. Take the cubic root, then the square root. Take the square root again to obtain approximately 1.0594631

To get a better sense of the concept of irrational numbers, ask students to use different approximations of $\sqrt[12]{2}$, for example 1.1, 1.06, 1.059, 1.0595. Then they should use these approximations to evaluate the frequencies of a given note starting with the *A*440 note. For example, what is the frequency for the *A* in the 6th octave when you multiply 440 Hz twelve times by each of the approximations. Remember that the A_6 is exactly 880 Hz. Students will derive a sequence of closer approximations like:

$440 \times (1.1)^{12} = 1380.91$ Hz
$440 \times (1.06)^{12} = 885.36$ Hz

but never exactly 880 Hz.

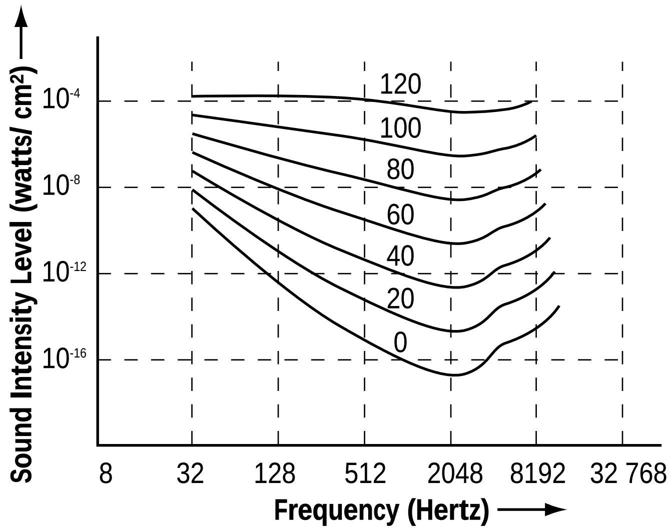
Activity 2.4: Decibels and Intensity of Sound

In this activity, students develop and use a formula to re-scale the units used in measuring sound intensity. This activity reinforces their ability to work with powers and with formulae involving powers.

Procedure:

On the overhead projector, superimpose a typical FRC (for example, Figure 11: Typical Frequency Response Curve) onto Figure 13: Levels of Sound Perception. Ensure that the frequencies of the scales in the two figures are enlarged to the same size. That is, the scales of the frequencies are the same in the two figures but the scales of the sound intensity are different, one is expressed in dB, the other in watts/cm^2)

Figure 13: Levels of Sound Perception



Tell students that when a sound travels through any medium, it carries energy. The amount of energy however is very small. For example, a full capacity stadium cheering loudly at a football game generates barely enough energy to warm one cup of tea.

The intensity of a sound is expressed in watts per square centimetre (or energy/sec/cm^2). That is, the intensity of sound is a power per unit area or an average rate of energy per unit area.

The ear is a very complex organ. It can perceive sounds once a certain intensity threshold is reached. Beyond a certain intensity, however, pain and damage to the ear drum result. Above and below this threshold, the ear cannot perceive any sound. The ear is also selective with respect to the frequency of the sound. The human ear can detect sounds that have a frequency between 20 Hz and 20 000 Hz. Outside this range, the sound cannot be perceived.

Referring to Figure 13: Levels of Sound Perception, discuss with students the features of these curves in relation to the FRC. Some aspects that can be discussed include:

- the threshold of hearing curve given for the intensity of each frequency
- the threshold of pain curve given for the intensity of each frequency
- the curves intersect and define a field where the sound is audible
- the scale of the frequencies is not linear (actually, it is a logarithmic scale) in order to cover a wide range of frequencies

Because of the human ear's extraordinary range of sensitivity (a range of over 1000 million million to 1), it is necessary to change the units of the intensity scale. The unit is called the decibel (after the Canadian inventor of the telephone, Alexander Graham Bell). The zero of the scale is 10^{-16} watts/cm^2.

This corresponds to the hearing threshold for the frequency 1 000 Hz and the intensity is then defined by the relation

$$I = 10^{-16} \times 10^{N/10} \text{ watts/cm}^2$$

N is called the number of decibels.

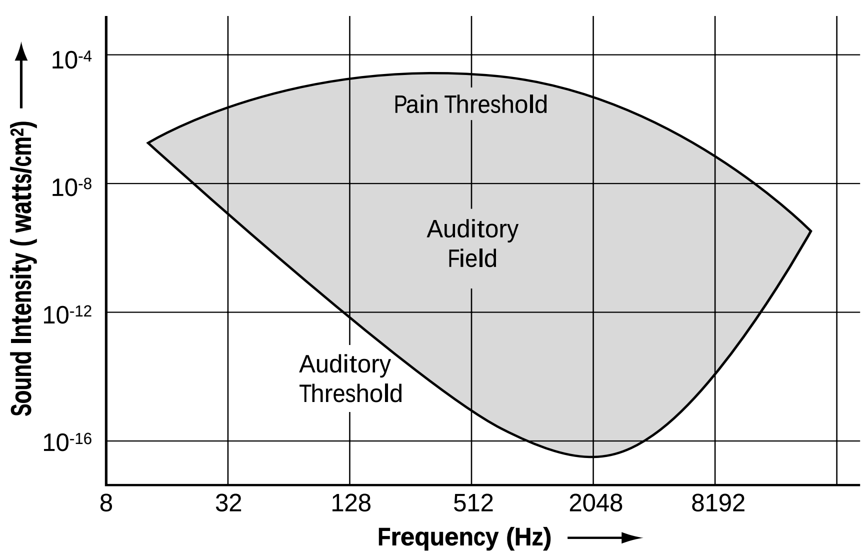
Have students use this formula to evaluate the intensities corresponding to 0 dB, 1 dB, 10 dB and 100 dB. Have students then relate these numbers to common sounds. Ask students to determine the number of dB corresponding to 10^{-13} (the sound of a whisper), 10^{-8} (a street with heavy traffic), and 10^{-1} watt/cm^2 (a jet plane with an afterburner).

Have students examine and complete Table 17: Sound Intensity Produced by Several Different Sources, in Student Activity: Speakers. Remind students to be careful about the units.

By examining the Figure 16: Levels of Sound Perception, students should note that the hearing threshold is not the same for all frequencies. As an extension, the students may examine and comment on the graph (Figure 14: Sensitivity of the Human Ear) of the intensity against frequency. Since the ear is not equally sensitive for all frequencies, the hearing threshold is not the same for all frequencies.

Have students compare this graph to the FRC so that they can determine the qualities of a stereo speaker and identify the stereo speaker performance that they cannot perceive.

Figure 14: Sensitivity of the Human Ear



Student Activity: Speakers

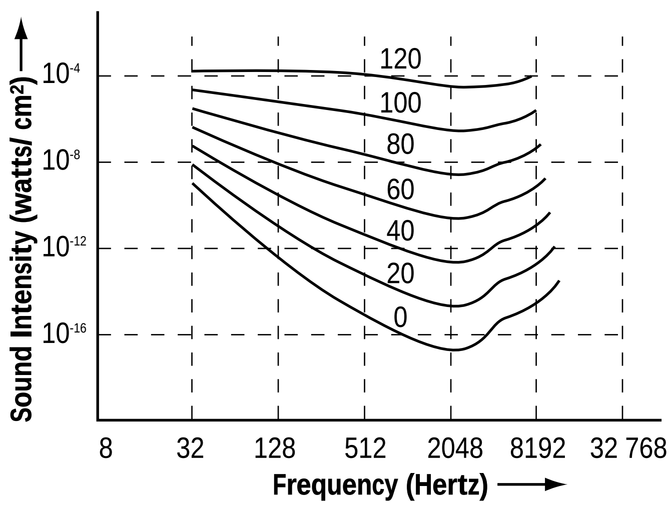
Student Name: ______________________ Date: ______________

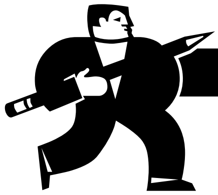
Table 15

Speaker Specifications

Speaker Model	Power Handling (W)	Maximum Power (W)	Frequency Range (Hz)	Plateau Height (dB)	Plateau Description	Price ($)

Figure 15: Levels of Sound Perception





Student Activity: Range of Emitted and Perceived Sounds

Student Name: ____________________________ Date: ___________________

Figure 16: Range of Emitted Sounds

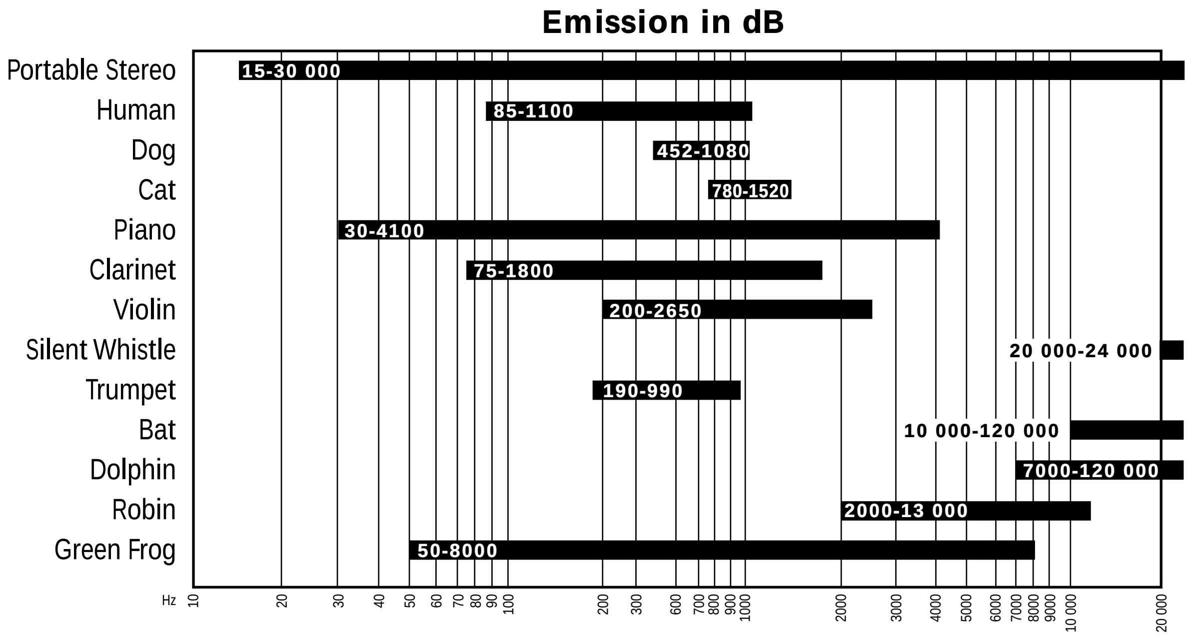
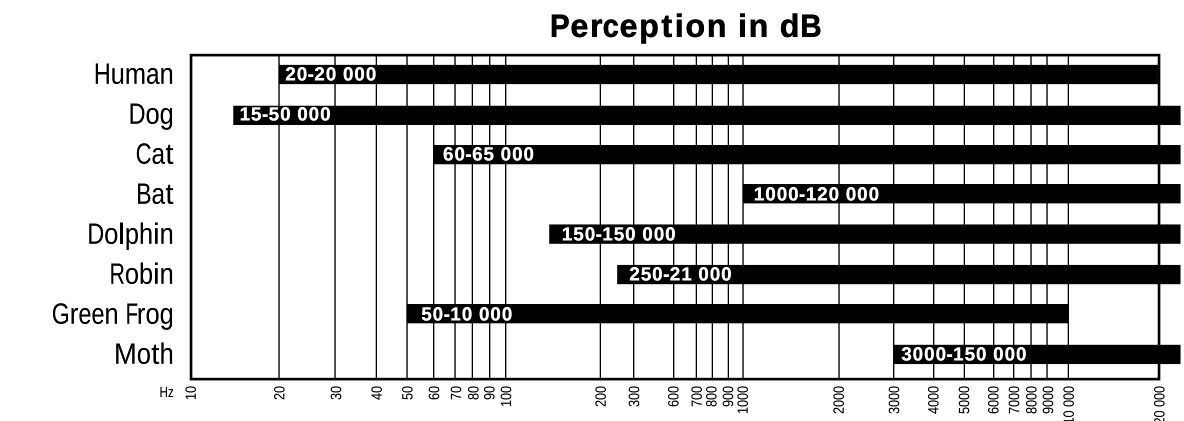
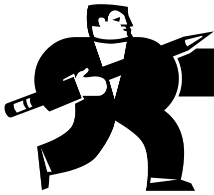


Figure 17: Range of Perceived Sounds



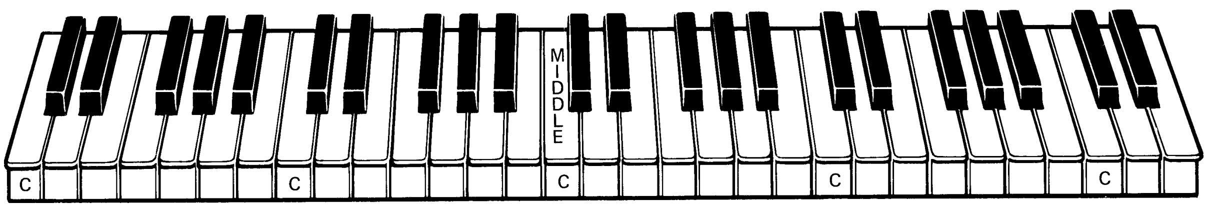


Student Activity: Tuning an Instrument

Student Name: ____________________________ Date: ___________________

On the following keyboard, identify the middle *C* note, that is, the *C* note in the middle of the keyboard.

Figure 18: Keyboard



Going to the left, identify the first *A* note - this is the *A*440 that is, the note that is supposed to be tuned exactly at a frequency of 440 Hz.

Identify all the notes for an entire octave (from *C* to *B* including the black notes).

Find all the *A* notes on the keyboard. What are the frequencies of these notes in Hz?

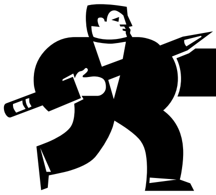
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
Hz:										

On a piano keyboard what is the frequency of the first possible audible *A* note from the left?

What is the frequency of the last possible audible *A* note to the right?

The first possible audible note produced by a church organ is a *C*, and the last possible note produced is also a *C*. How many notes are there on a keyboard? How many octaves? How many notes are there per octave?

From the *A*440, the next note to the right is an *A#*. You need to find the frequency of this note. Propose a strategy for doing this.



Student Activity: Tuning an Instrument

Use your result to estimate the frequency of middle *C*.

The frequency you have determined above is called an irrational number. A number that cannot be expressed as a fraction is an irrational number.

Use your calculator to determine the 4th root of 2

Use your calculator to determine the 8th root of 2

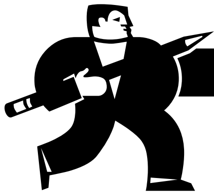
Use your calculator to determine the 6th root of 2

Use your calculator to determine the 12th root of 2

Use your calculator to determine the 24th root of 2

Numbers like $\sqrt{2}$ and π are irrational numbers and can only be approximated. Any square root of a number that is not a perfect square is an irrational number. You can never find an exact value for these numbers-only a close approximation. The approximation you determine depends on the calculator you use.

To convince yourself, do the following: use your calculator and take the square root of 2 twenty times. Then, use the square function and go back twenty times. Complete Table 16: Demonstrating the Limits of a Calculator, with your results. Compare your results with those of your classmates who have different models of calculators. Comment on the accuracy of the approximations.



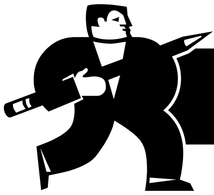
Student Activity: Tuning an Instrument

Table 16

Demonstrating the Limits of a Calculator

#	Number	$\sqrt{\ }$	Square
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

In what ways are the approximations you are able to obtain from your calculator good enough for most applications?



Student Activity: Sound Intensity Produced by Several Different Sources

Student Name: ____________________________ Date: ___________________

Complete Table 17: Sound Intensity Produced by Several Different Sources.

Table 17

Sound Intensity Produced by Several Different Sources

Source	dB	watt/m^2
Threshold of hearing	0	
Normal breathing	10	10^{-11}
Whispering (2 m)		10^{-10}
Quiet restaurant	50	
Normal conversation between two people		10^{-6}
Vacuum cleaner	80	
Loud music in an average room		10^{-3}
Rock concert	110	
Commercial airplane at takeoff	120	
Pain threshold	130	
Military airplane at takeoff		10^{2}
Rocket takeoff		10^{3}
Instant perforation of the ear drum	160	

Activity 3: Beehives

Objective :

Students realize that bees are able to use a given space in an optimal way. This is an optimization problem where a space is used to stock a maximum amount of a material, that is honey.

This activity is divided into five parts leading students progressively to understand a mathematical model that helps explain an optimization problem.

Activity 3.1: Concentric Squares
Students describe a pattern, represent it geometrically and then symbolically from a numerical representation.

Activity 3.2: Geometric Figures
Students review the properties of regular geometric figures.

Activity 3.3: Beehive
Students use their knowledge of the properties of geometric figures and their ability to represent a pattern in the construction of a cardboard beehive.

Activity 3.4: The Meeting
Students use their cardboard model to create a game that can be used to reinforce their knowledge of geometrical transformations in a plane.

Activity 3.5: Hexagonal Behive Cells
Students perform calculations on perimeter, area and volume to determine the optimal figure than can be used to construct a model.

Activity 3.1: Concentric Squares

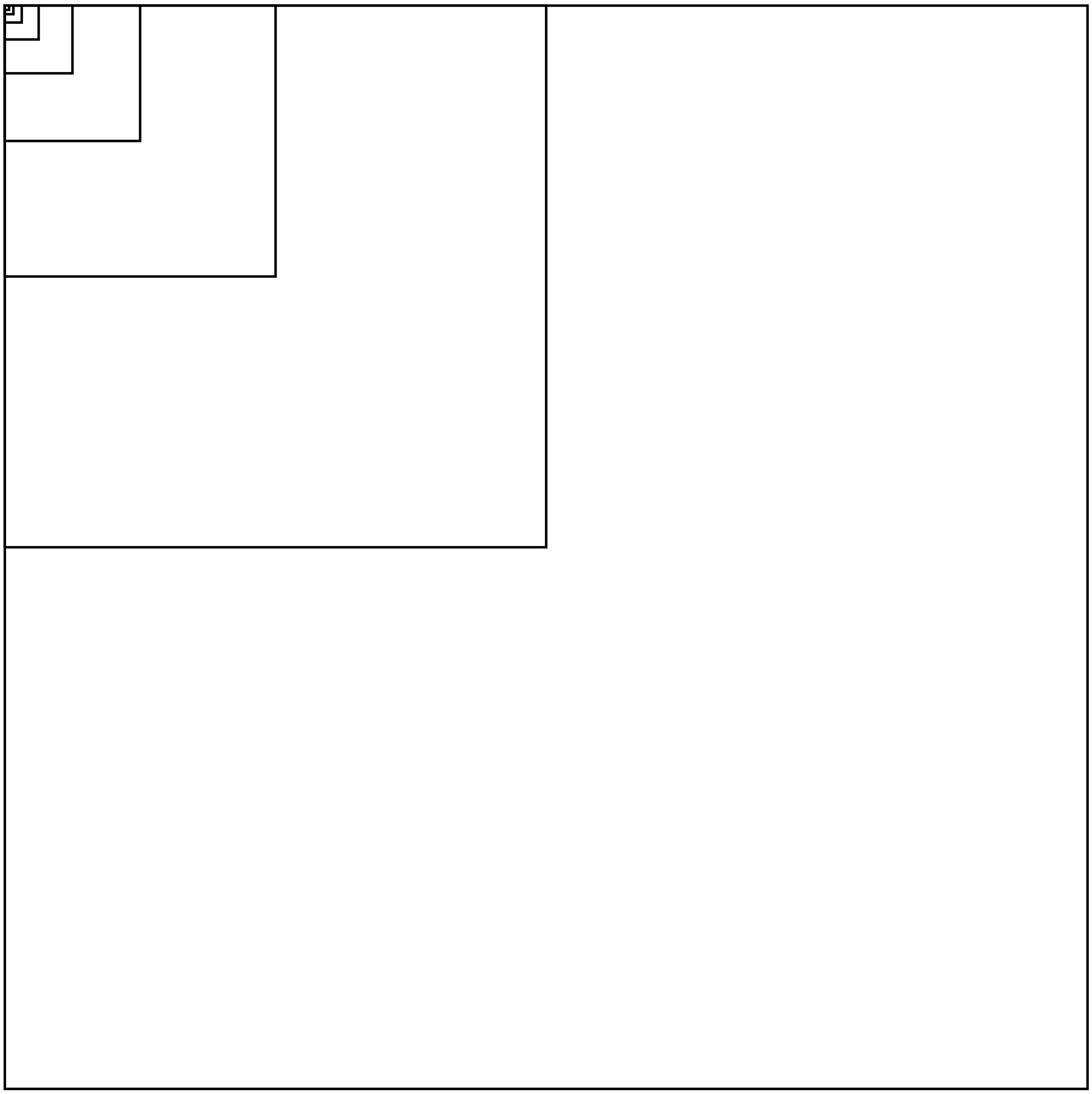
Objective:

The objective of this activity is to construct a series of concentric squares from a given square by halving the sides. The objective is to represent the recursive property geometrically and then algebraically.

Procedure:

Students construct a square of side 20 cm by using a square and a pair of compasses. See Figure 19: Concentric Squares below.

Figure 19: Concentric Squares



Students complete Table 18: Rules for Concentric Squares, and answer the questions in Student Activity: Concentric Squares.

Table 18

Rules for Concentric Squares

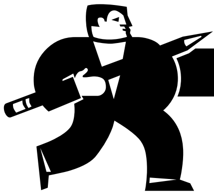
Steps	# Squares	Total # Squares	Rule
1	1	1	1 x 1 = 1
2	4	5	2 x 2 = 4 and 4 + 1 = 5
3	9	14	3 x 3 = 9 and 9 + 5 = 14
4	16		4 x 4 = 16 and 16 + 14 = 30 or 4 x 4 + 3 x 3 + 2 x 2 + 1 x 1 = 30
5	25	55	5 x 5 + 4 x 4 + 3 x 3 + 2 x 2+ 1 x 1 = 25 + 30 = 55
6			
7			
8			
9			
10			
11			
12			
20			20 x 20 + ... + 1 x 1 = 6 998 670
50			
100			
1000			

Have students compare the different ways their calculators operate when they evaluate the expression below.

$(n\ x^{2}+)$: $20x^2 + 19x^2 + ... = 6\ 998\ 670$

Have students verbally express the rule, graph the results and finally express the rule algebraically.

If n is the number of steps, the number of squares is $n^2 + (n\text{-}1)^2 + ...+1$



Student Activity: Concentric Squares

Student Name: ______________________ Date: ______________

Procedure:

- draw a square with 20 cm sides
- divide one side into 2
- construct a square inside the big square (this smaller square will have 1/4 the area of the original square with 20 cm sides)
- divide one side into 2
- construct a square inside the second square (this smaller square will have 1/16 the area of the original square with 20 cm sides)

Table 19

Rules for Concentric Squares

Steps	# Squares	Total # Squares	Rule
1	1	1	
2	4	5	
3	9	14	
4	16		
5			
6			
7			
8			
9			
10			
11			
12			
20			
50			
100			
1000			

Student Activity: Concentric Squares

Answer the following questions:

How do you obtain the numbers in column 2?

How do you obtain the numbers in column 3?

What is the number corresponding to 20 steps?

What is the number corresponding to 1000 steps?

Express the rule in your own words.

Express the rule in algebraic terms.

Activity 3.2: Geometric Figures

Objective:

The objective of this activity is to review the properties of basic geometric figures. Students use a pair of compasses to construct the basic regular polygons. In particular, they review concepts related to the angles and symmetries of basic regular polygons.

Procedure:

Make sure that students know how to use their compasses properly.

The first polygon is the hexagon that is obtained by taking the radius as the side. Ask students to compare this procedure to other ways of using a ruler and protractor. Refer to Project A for more details.

Ask students to draw the axes and to verify their accuracy by folding along the axes of symmetry.

For the regular dodecagon, use the compass to divide one side of the hexagon into two equal parts and join these together.

To obtain a square, join the vertices of the dodecagon three by three.

To obtain an equilateral triangle, join the vertices of the hexagon two by two, and so on.

Have students measure the sides of the polygon by using a pair of compasses. They should complete the Table 22: Geometric Figures in Student Activity: Geometric Figures. Ask students to discuss why this method is more accurate than a direct measurement with a ruler. See Table 20: Regular Polygons (Radius of Circle: 10 cm).

Have students extrapolate the perimeter of a circle by constructing a rule from the pattern observed in the Table 20: Regular Polygons (Radius of Circle: 10 cm). They should represent their rule in the form of a graph on their graphing calculator.

Students should realize that it now possible to construct a pentagon with this method. Ask students to design a method for constructing a pentagon. Have them draw and count the axes of symmetry and then enter this number in Table 21: Axes of Symmetry. Ask students to measure the angles of the pentagon and to record the values in Table 20: Regular Polygons. They may create a recursive rule for the angles in the same way. Have students compare the relationship between the number of sides and the angle values.

Table 20

Regular Polygons (Radius of Circle: 10 cm)

# Sides	Name	# Vertices	Length of Side (cm)	Perimeter (cm)
3	triangle	3	17.3	17.3 x 3 = 51.9
4	square	4	see calculation below	
6	hexagon	6	10	
8	octagon	8		
12	dodecagon	12	see calculation below	
21	?	24		

Note: Length of the sides of regular polygons

The angles in a regular polygon are all equal: multiply 180^{o} by the number of sides minus 2 and divide by the number of sides, for example:

(3 sides - 2) x 180^{o}/3 sides = (1) x 180^{o}/3= 60^{o}

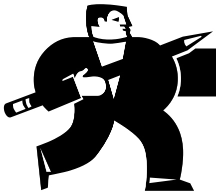
equilateral triangle: 1 x180^{o} /3 =60^{o}
square: 2 x 180^{o}/4 = 90^{o}
hexagon: 4 x 180^{o}/6 = 120^{o}
octagon: 6 x 180^{o}/8 = 135^{o}
dodecagon: 10 x 180^{o}/12 = 150^{o}

side of hexagon: length of side is given as 10 cm
half side of triangle: side of hexagon x 0.86 = 8.6 cm
side of dodecagon: half side of hexagon divided by 0.96 = 5/0.96 = 5.2 cm

Table 21

Axes of Symmetry

Name of Shape	Number of Vertices	Number of Axes
triangle	3	3
square		



Student Activity: Geometric Figures

Student Name: ____________________________ Date: ___________________

Draw a circle of radius x cm larger than 10 cm.
Construct a hexagon, a triangle, and so on by following your teacher's instructions.
At each step, name the polygon, count the number of sides and measure the side lengths and the angles. Enter your results below in Table 22: Geometric Figures:

Table 22

Geometric Figures

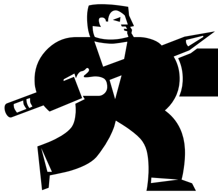
# Sides	Name	# Vertices	Side Length (cm)	Interior Angle (degrees)	Perimeter (cm)	Rule
3						
4						
6						
8						
12						

Propose a method to construct a pentagon.

Propose a method to construct a dodecagon.

What is the length of the pentagon side?

What is the length of the dodecagon side?



Student Activity: Geometric Figures

What is the perimeter of a regular 24-sided polygon?

What is the perimeter of a regular 36-sided polygon?

What is the perimeter of a regular 216-sided polygon?

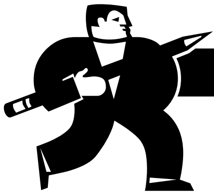
What is the perimeter of a regular 1296-sided polygon?

Draw all the axes of symmetry and complete Table 23: Number of Axes of Symmetry.

Table 23

Number of Axes of Symmetry

Name of Polygon	Number of Vertices	Number of Axes of Symmetry
triangle	3	3



Student Activity: Geometric Figures

For each polygon, count the number of angles, add all the angles, measure the angles and complete Table 24: Polygon Angles.

Table 24

Polygon Angles

# Sides	# Angles	Angle Measurement	Rule
3			
4			
5			
6			
8			
10			
12			

What is the interior angle of a 9-sided polygon?

What is the interior angle of a 27-sided polygon?

What is the interior angle of a 243-sided polygon?

What is the interior angle of a 729-sided polygon?

Explain and apply the rule you have derived.

Activity 3.3: Beehive

Objective:

In this activity, students conceptualize the properties of geometric figures and apply different recursive rules to fill a square of given dimensions.

Procedure:

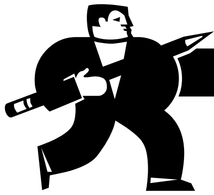
Have students draw and cut out a hexagon with a side of 5 cm using the method described earlier. They will use this model for drawing all the cells of the beehive.

Have students determine the center of a square piece of cardboard (50cm x 50cm) and then draw the two diagonals. Placing the model at the centre, they start tracing the beehive following the instructions, i.e., forming concentric rings of hexagons.

After two rings, students are asked to draw the axes of symmetry:
- one vertical through the centre
- one horizontal through the centre
- one axis at 30˚
- one axis at 60˚.

The axes will be used in the next activity, The Meeting. See Figure 23: Game Board.

Students complete the tables and the graphs in Student Activity: Construction of the Beehive, and answer the questions.



Student Activity: Construction of the Beehive

Student Name: ____________________________ Date: ___________________

Materials:

- sheet of solid cardboard
- scissors
- ruler
- pair of compasses

You are asked to:

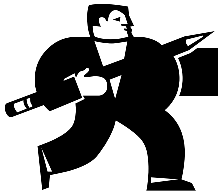
- Follow the instructions to construct the cardboard model of a beehive.
- Construct a table like Table 25: Total Cell Parameters to enter your data.
- After each ring use your own words to express the rules for calculating the number of cells, the total number of cells, and the perimeter.
- Represent on a graph how these quantities vary with the number of rings.
- Derive an equation of a relationship between the number of cells and the number of rings.
- Elaborate on the following questions:

What are some of the things that could happen if bees constructed a beehive with octagonal cells?

What are some things that could happen if bees constructed a beehive with pentagonal cells?

What are some things that could happen if bees constructed a beehive with square cells?

What are some reasons for the cells being hexagonal in form?



Student Activity: Construction of the Beehive

Table 25

Total Cell Parameters

Ring Number	Number of Cells Per Ring	Total Number of Cells	Number of Sides Per Ring	Total Number of Sides
1	1	1	6	6
2	6	7	24	30
3	12	19		
4	18			
5				
6				
12				
36				
100				
RULE				

Explain how you derived the expression for the rules.

How long does it take to fill the square if one ring is constructed every hour? Show your work.

How long does it take to finish the job if one cell is constructed every 10 minutes?

Estimate the actual number of cells in a beehive. Describe the process you used.

Student Activity: Construction of the Beehive

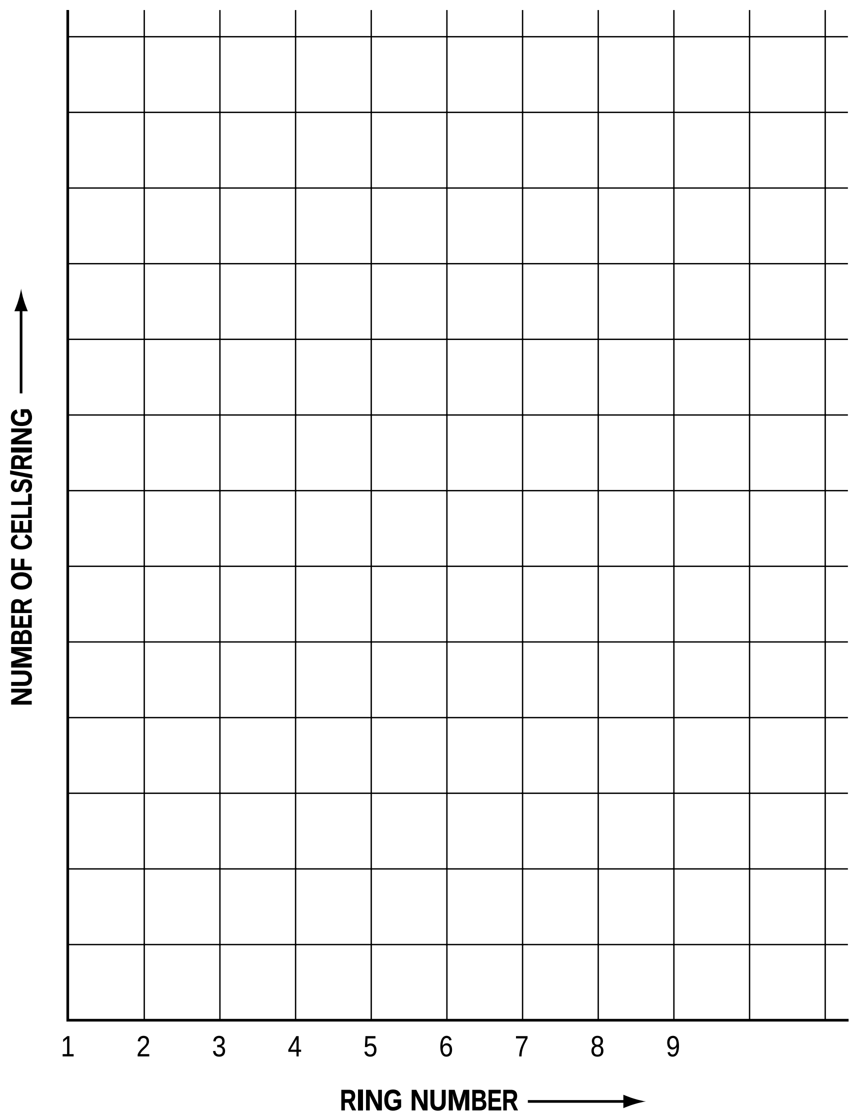
Estimate the dimensions of a real cell (side length, area, volume). Describe the process you used.

What is the total perimeter of the beehive?

Use the following blank graph to represent the number of cells, the total number of cells and the perimeter per ring. Use an *x* to plot the intersection of the number of cells and the ring number.

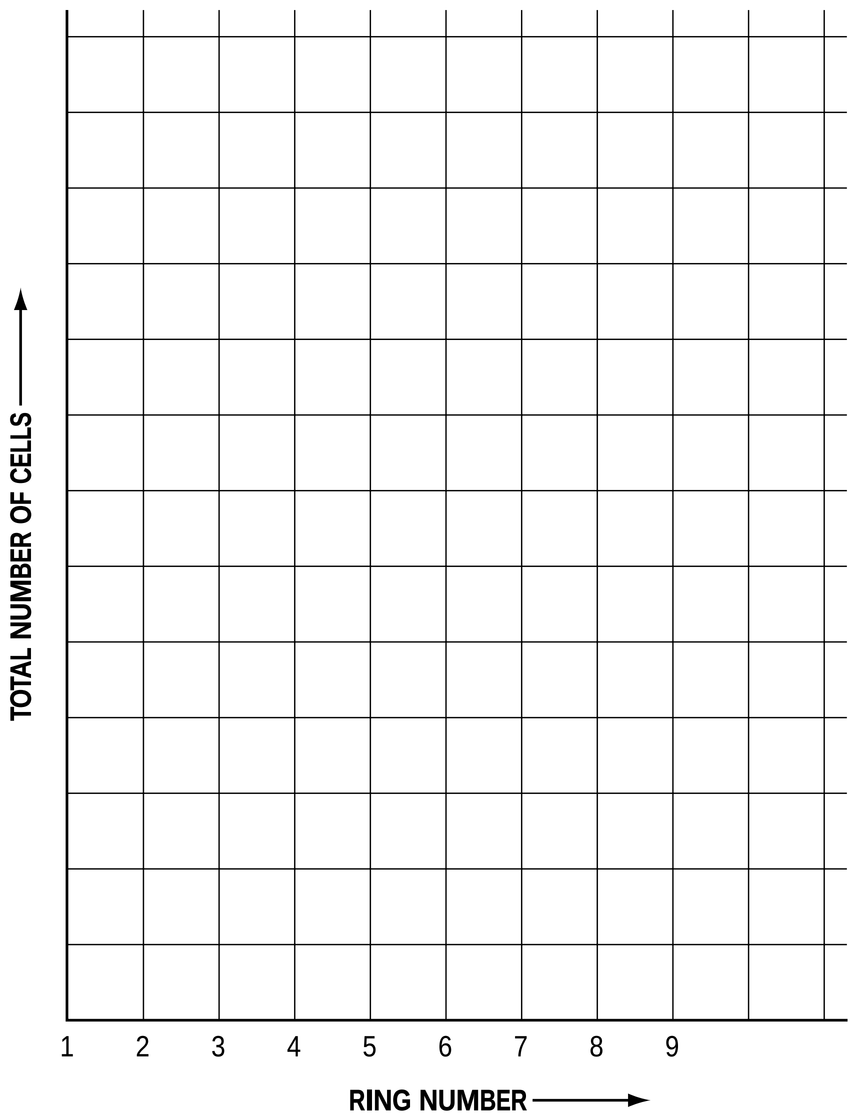
Student Activity: Construction of the Beehive

Figure 20: Number of Cells Per Ring



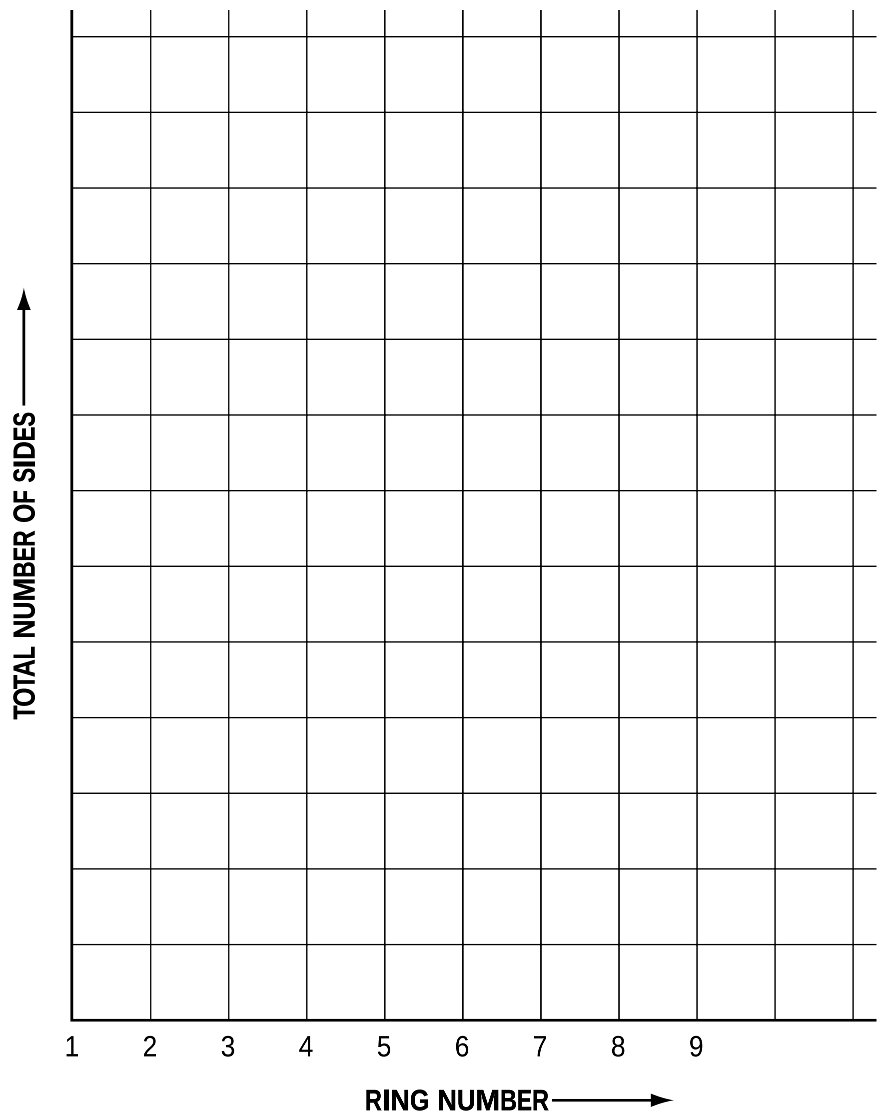
Student Activity: Construction of the Beehive

Figure 21: Total Number of Cells



Student Activity: Construction of the Beehive

Figure 22: Total Perimeter of Beehive



Activity 3.4: The Meeting

Objective:

The objective of this activity is a review of the concepts of displacement in the vertical, horizontal and diagonal planes.

Procedure:

The object of the game is to meet with a partner sitting in a given cell. Essentially students are playing the traditional naval combat game and the same rules apply as in the regular game.

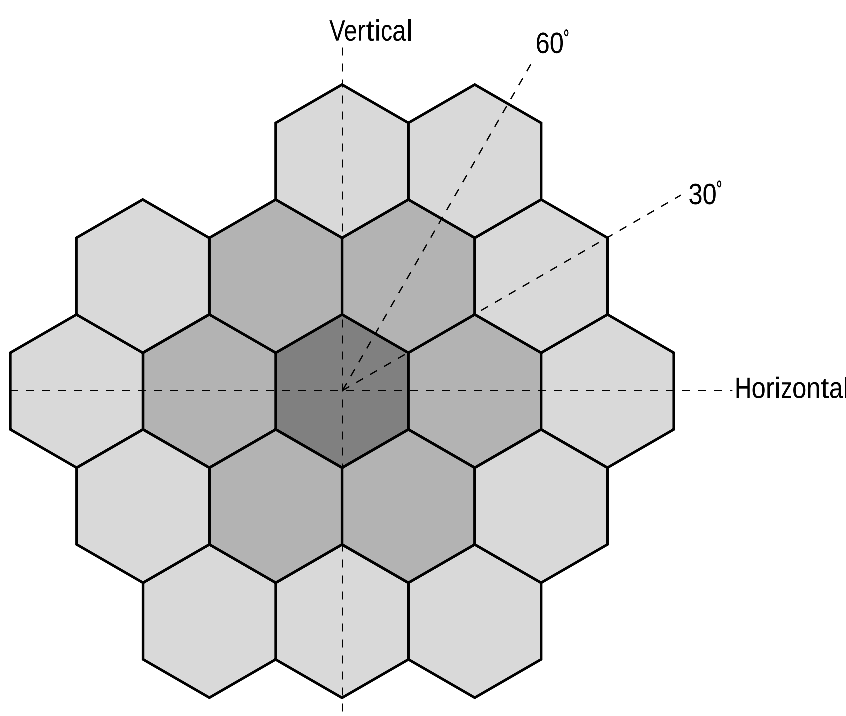
Constructing the Game Board

Have each student build their own game board. Instead of drawing a grid of squares on the game board, students fill the game board with hexagons. All students should have the same size hexagons on their game board and these should all be identified in the same way. The following directional labels should be added to the perimeter of each game board:

- horizontal
- vertical
- 30°
- 60°

The diagram below, Figure 23: Game Board, provides a guide of what the game board should look like.

Figure 23 : Game Board



Students can choose any game pieces they wish to mark their meeting place, for example chess pieces or checkers.

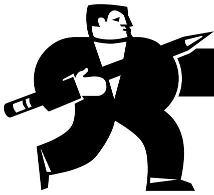
Finally, each game board must have a vertical backboard divider so that students cannot see the hexagon their opponent has identified as his meeting place.

Playing the Game

The objective of the game is for a player to reach his opponent's meeting place. The player who reaches the meeting place using the fewest number of moves is the winner.

Discuss and establish the game rules as a class

Students throw dice to determine the number of cells and the direction of movement. They describe their move to their opponent, for example: "I am moving left at a diagonal of 30˚, 6 cells from centre."



Student Activity: The Meeting

Student Name: ______________________ Date: ________________

Materials:

- game board with hexagonal spaces
- dice
- queen and a king from a chess game, or other game pieces

Procedure:

The objective of the game is for a player to reach her opponent's meeting place. The player who reaches the meeting place using the fewest number of moves is the winner.

Discuss and establish the game rules as a class.
Students throw dice to determine the number of cells and the direction of movement. They describe their move to their opponent, for example: "I am moving left at a diagonal of 30˚, 6 cells from centre."

Activity 3.5: Hexagonal Beehive Cells

Objective:

Why are beehive cells hexagonal? The objective of this activity is for students to measure length, perimeter and area to determine the optimal ratio, perimeter/area, for different tiling of a square. They derive a formula giving the ratio as a function of the number of sides for the regular polygons.

Procedure:

Have students draw and cut out models of different regular polygons (e.g., squares, pentagons, hexagons, octagons) all of them are inscribed in the same circle having a radius of 5 cm.

Ask them to derive the formula for the total area and the total perimeter in the same way as they did for the beehive problem. Prior to deriving the formula, ask them to graph the results.

Ask students to comment on the way the variables are increasing.

Ask students to complete Table 25: Total Cell Parameters and Figure 24: Generic Graph.

Ask students to extrapolate for polygons with 10, 12, and more sides.

Ask them what would happen if they were trying to tile the area with circles.

Ask students to discuss why hexagons are used in nature in many different instances.

Student Activity: Why are Beehive Cells Hexagonal?

Student Name: ______________________________ Date: ____________________

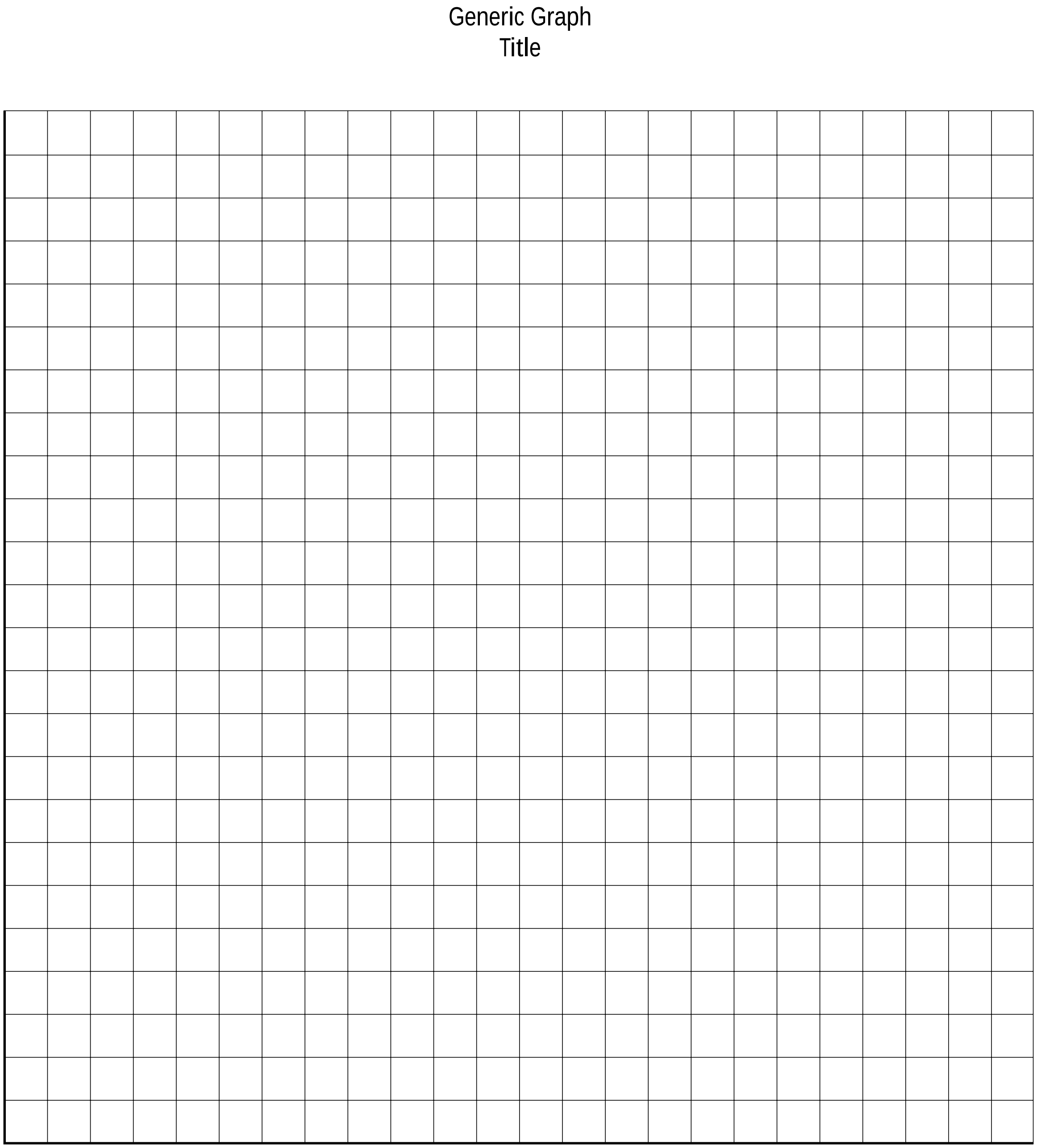
Materials:

- cardboard
- models of hexagons, squares, octagons and pentagons
- scissors

Procedure:

- Draw and cut out models of different regular polygons (e.g., squares, pentagons, hexagons, octagons) all of them inscribed in the same circle having a radius of 5 cm.
- Derive the formula for the total area and the total perimeter in the same way as you did for the beehive problem. Record your information using tables like the ones previously used to record your data.
- Use graphing software to draw the graphs in each situation and recopy on a blank graph like the one that follows.

Figure 24: Generic Graph



Phase III: Iso- and Geosynchronous Orbits

Activity 1: Achilles and the Turtle

Activity 2: Planets

Activity 3: Andromeda and the Liquid-Mirror Telescope

Activity 4: Big Bang

Activity 5: Autopsy of a Collision

Phase Synopsis:

In this evaluation phase students are provided with a number of opportunities to demonstrate the skills and knowledge they have acquired during the project. They are presented with real-world situations and must develop models with formulae containing powers and perform operations with a technological aid.

The objectives of this phase are:
- to increase student awareness of the necessity of modeling situations with formulae involving powers
- to reinforce student skills, and abilities to manipulate algebraic expressions containing powers with integral exponents and rational bases by using the Laws of Exponents
- to increase student skills in correctly using calculators with very large and very small numbers

As the teacher, you will need to determine the extent to which you will guide and monitor the activities. You may choose to use the flowchart of mathematical modelling as a guideline. At this point in the project, students have developed a sufficient number of concepts, and have the ability to select and conduct an activity of their choice. Five suggestions follow for such a problem-based learning activity.

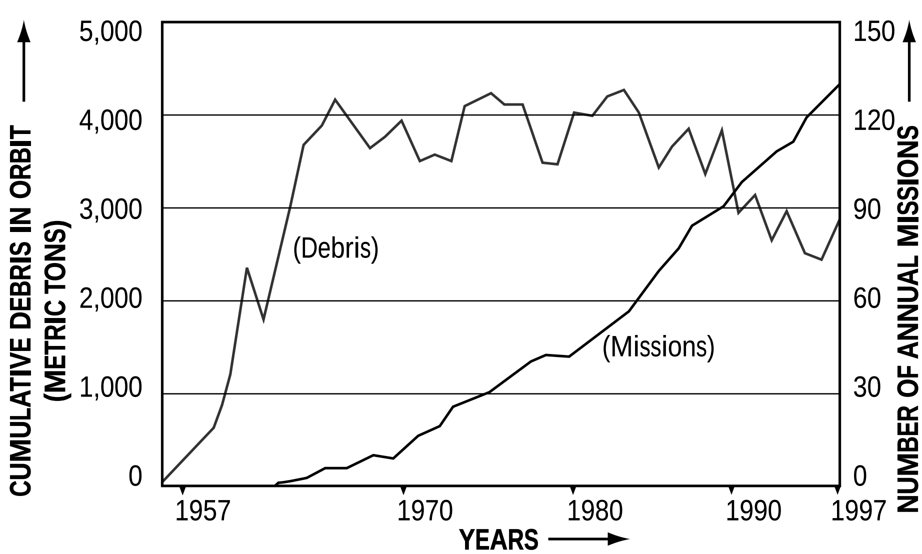
Students are asked to use the Internet to find necessary data and information, and to present their final report by making intensive use of new technology. The final report could be a class presentation or the preparation of an article for the school newspaper.

Provide students with the following four tables of data and three graphs. Offer challenges such as the following to assess their understanding of exponents, scientific notation and the Laws of Exponents:

- What is the ratio of the aphelions of Pluto and Earth?
- What is the volume of Venus?
- What is the mass of the Earth and the Moon combined?
- What is the difference between the period of revolution of the Earth and that of Neptune?
- Express the Earth's rotational velocity at the equator in km per year.
- Uranus rotates in the opposite direction to that of the Earth/Sun system. Explain.
- Estimate how long it takes for a fixed object on Uranus to be seen again from a fixed point on Earth.
- Estimate the ratios of the gravity at the equator for each planet with respect to Earth.
- Explain what escape velocity means. Calculate the ratio of the escape velocity on Icarus to that of Earth.
- Analyse and comment upon Figure 25: Cumulative Mass of Debris, and Figure 26: Earth Satellite Population.
- Define Solar Ratio Flux.

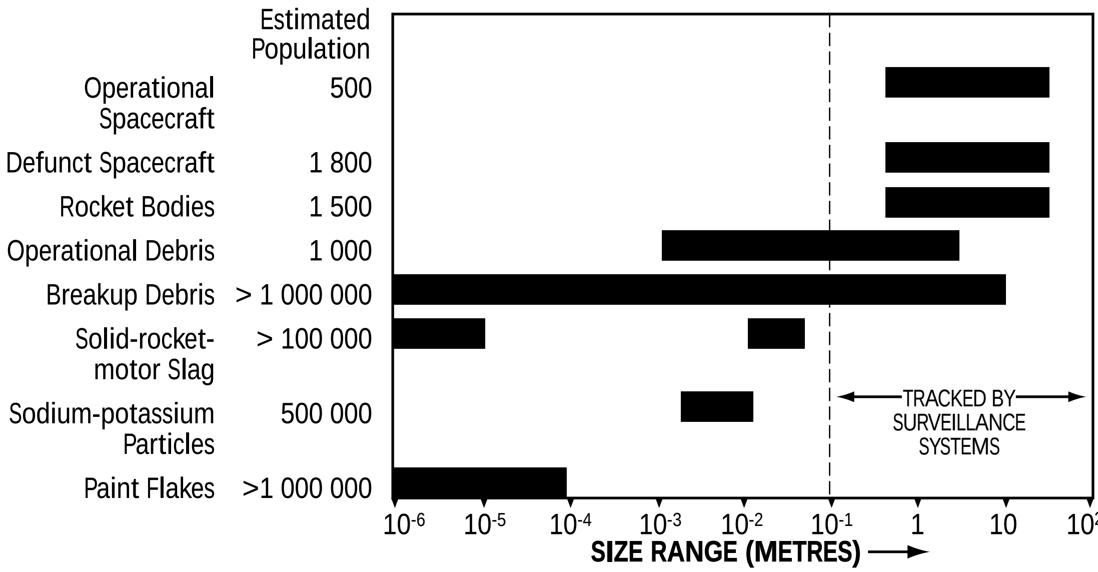
Using the information in Figure 27: Spatial Density, estimate the mass of satellites and debris at altitudes of 20 000 km and 40 000 km.

Figure 25: Cumulative Mass of Debris



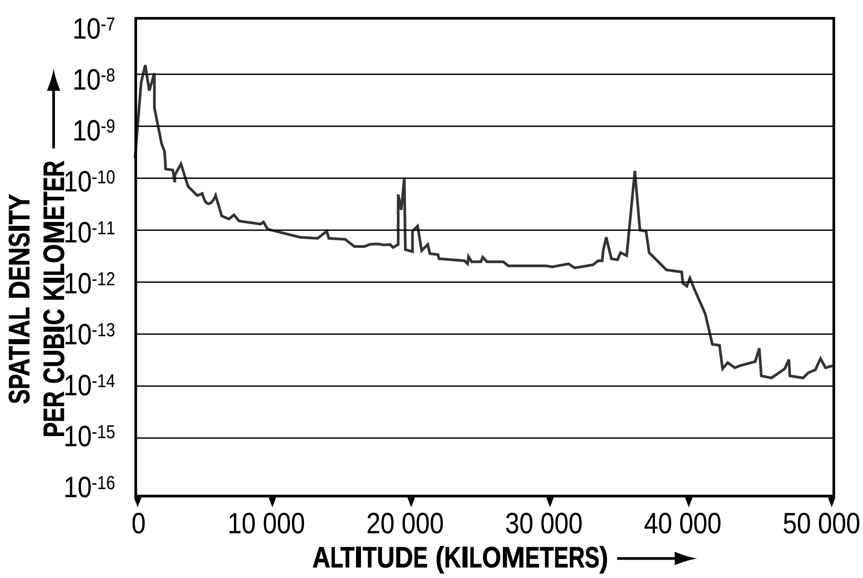
Adapted from: Nicholas L. Johnson. "Monitoring and Controlling Debris in Space", *Scientific American* (August 1998): 62-67.

Figure 26: Earth Satellite Population



Adapted from: Nicholas L. Johnson. “Monitoring and Controlling Debris in Space”, *Scientific American* (August 1998): 62-67.

Figure 27: Spatial Density



Adapted from: Nicholas L. Johnson. “Monitoring and Controlling Debris in Space”, *Scientific American* (August 1998): 62-67.

Activity 1: Achilles and the Turtle (1 hour)

Objective:

Using the concepts related to rational numbers and their limitations, students reproduce the experiments of Zeno in different contexts. In reproducing the experiments they discover a paradox of the Ancient Greeks. They realize the limits of the set of rational numbers.

Project product: Report on the concept of limit.

A. Situation

B. Variables

C. Assumptions

D. Model

E. Solving the Model

F. Testing the Model

G. Simplifying the Model

H. Refining the Model

I. Applying the Model

Activity 2: Planets (2 hours)

Objective:

Students develop and apply algebraic formulae in order to model the motion of planets and satellites (Kepler's Third Law).

Project product:

Mathematical models of the motion of planets and satellites.

Activity 3: Andromeda and the Liquid-Mirror Telescope

Objective:

From data obtained from the Dominion Astronomical Observatory and the Western Ontario Light Detection and Ranging Observatory, students perform measurements and calculations on galaxies.

Students propose ways to observe the sky and to calculate distances (or time) between stars.

Project product:

A report on different ways of observing the sky.

Activity 4: Big Bang

Objective:

Students evaluate their ability to use and apply expressions containing powers by exploring some aspects of the expansion of the Universe, the Theory of Relativity and Quantum Theory.

Project product:

Report on the various interpretations of the origin and the expansion of the Universe.

Activity 5: Autopsy of a Collision

Objective:

The objective of this activity is for students to evaluate the probability of a collision between two objects in a sequence of increasingly complex settings. They relate the probability of collision to the density of objects revolving around in the same orbit.

Project product:

- Report describing the various factors involved in predicting a space collision and proposing possible solutions to the problem, or
- Presentation: Monitoring and Controlling Debris in Space
- Class presentation or preparation of a documented article in the school newsletter, or a video presentation.