PRINCIPLES OF MATHEMATICS 12

EXAMINATION SPECIFICATIONS

SEPTEMBER 2004

Assessment Department

The information in this booklet is intended to be helpful for both teachers and students. Teachers are encouraged to make this information available to all students.

PRINCIPLES OF MATHEMATICS 12

Summary of Major Changes to Principles of Mathematics 12 Specifications September 2004

There are two changes to the Principles of Mathematics 12 Specifications for the 2004/2005 school year affecting the 2005 Examinations.

1. The multiple-choice section of the examination (Part A) is now composed of 44 questions worth 1.5 marks each; the written-response section (Part B) is composed of 5 questions worth 24 marks for a total value of 90 marks for both sections.

2. The examination will contain a non-calculator active section in Part A. Administration of the examination will be as follows:

   At the beginning of the examination, students will receive the examination paper and two bubble sheets. Students will begin Part A using the first bubble sheet with an approved calculator placed on the floor below their desks.

   PART A: 44 Multiple-choice questions

   Section I: The first **16 questions** are to be done without the use of any calculator whatsoever. On the examination, the suggested time given for Section I will be **35 minutes**; however, students will be given 45 minutes before they must hand in the first bubble sheet. At the end of the 45 minutes, the first bubble sheets will be collected and students may then place their calculators on their desks and use their calculator for the remainder of the exam session. During the first 45 minutes, students may proceed onto other questions on the examination; however, no student will be able to access their calculator until all of the first bubble sheets have been collected at the end of 45 minutes. **There are many questions in the remaining part of the examination that are not calculator-active.** Once the first bubble sheet is handed in, students will not be able to go back to any of these first 16 questions.

   Section II: The remaining **28 questions** will contain **some** questions for which a graphing calculator is required. The suggested time given for Section II will be **55 minutes**.

   PART B: 5 Written-response questions

   There will be **some** questions for which a graphing calculator is required. The suggested time given for Part B will be **30 minutes**.

The suggested time for each part of the examination reflects the fact that the examination is designed to be completed in **two hours.** *Since students are allowed to take up to 30 minutes of additional time to finish, 10 minutes of this additional time has been given to Part A, Section I, of the examination.*

* For teacher information, an asterisk has been placed beside those Sample Questions in this document to indicate the type of question which might lend themselves best to the non-calculator section (Part A, Section I) of the examination.

Shaded text indicates either significant changes to the Examination Specifications document since the 2003 printing, or important new information about the subject.

The intent of the *Examination Specifications* is to convey to the classroom teacher and student how the Principles of Mathematics 12 curriculum will be tested on the provincial examinations. The Table of Specifications provides percentage weightings for each of the curriculum organizers as well as the cognitive levels that are applied to questions. A detailed description of examinable material within each curriculum organizer will be found in the curriculum section and Appendix A of the Principles of Mathematics 12 component of the *Mathematics 10 to 12 Integrated Resource Package 2000* (IRP #110).

It is essential that teachers thoroughly familiarize themselves with the content of the *Mathematics 10 to 12 Integrated Resource Package 2000* (IRP #110).

URL: http://www.bced.gov.bc.ca/irp_implementation/math1012.htm

Sample questions are provided in this resource.

Aside from an approved calculator, electronic devices, including dictionaries and pagers, are **not** permitted in the examination room.

It is expected that there will be a difference between school marks and provincial examination marks for individual students. Some students perform better on classroom tests and others on provincial examinations. School assessment measures performance on all curricular outcomes, whereas provincial examinations may only evaluate performance on a sample of these outcomes.

The provincial examination represents 40% of the student's final letter grade and the classroom mark represents 60%.

Mathematical justification for changes:

The intention of creating a non-calculator component of the examination is so that particular prescribed learning outcomes (PLOs) can be tested as they were intended. The multiple-choice format can at times compromise the intent of a question if a calculator is available. For example, it is important for a student to know exact values of trigonometric functions (such as $\sec\frac{2\pi}{3}$) and not be able to use a calculator to check this information. It is also important for students to be able to solve an equation algebraically and not simply substitute distracters into the equation to check their accuracy numerically. A student must also know domain and range restrictions or characteristics of a function (such as $y = \log(2x-5)$) without the ability to check this feature graphically.

It is important to stress that students continue to view different concepts in the classroom with a multiple-representational approach; that is, they should view mathematical ideas analytically, graphically, numerically and verbally. It is also important that students understand the full implications of different algebraic features and be able to demonstrate their understanding in a calculator-free environment and not use the calculator to subvert understanding of critical ideas.

Acknowledgement
The Assessment Department wishes to acknowledge the contribution of British Columbia teachers in the preparation and review of this document.

Table of Contents

PRINCIPLES OF MATHEMATICS

CALCULATOR POLICY

Advances in technology have made a profound impact on the teaching of mathematics; in particular, graphing calculators and computer programs have enhanced the delivery of mathematics at all levels. The Principles of Mathematics 12 course encourages teachers to integrate the use of such technology into their lessons and to investigate the added depth that results from exposing students to a variety of methods of solution. Problem solving, which is central to the curriculum, will be strengthened by creative use of these powerful tools. Students will learn both the advantages and the limitations of graphical solutions to problems as well as the use of technology to develop their own analysis of situations.

The *Mathematics 10 to 12 Integrated Resource Package 2000* (IRP #110) states that new technology has changed the level of sophistication of mathematical problems encountered today, as well as the methods that mathematicians use to investigate them. Graphing tools, such as computers and calculators, are powerful aids to problem solving. The power to compute rapidly and to graph mathematical relationships instantly can help students explore many mathematical concepts and relationships in greater depth. When students have the opportunity to use technology, their growing curiosity can lead to richer mathematical invention.

Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM) and the British Columbia Association of Mathematics Teachers (BCAMT) have strongly endorsed the use of graphing calculators in mathematical instruction. They also emphasize that if calculator and computer technologies are now to be accepted as part of the environment in which students learn and do mathematics, such tools should also be available to students in most assessment situations.

As a result, the calculator policy for the Principles of Mathematics 12 Provincial Examination has been developed to include the use of graphing calculators. Although many questions will not require the use of a graphing calculator, there could be some questions where the calculator capabilities could be used.

If, in a justification, the student refers to information produced by the graphing calculator, this information must be presented clearly in the response. For example, if a graph is used in the solution of the problem it is important to sketch the graph, showing its general shape and indicating the appropriate values. If the statistical features of the calculator are used, it is important to show the function with the substitution of the relevant numbers. For example: in part of the solution it is acceptable to show normalcdf (10, 40, 50, 20) or the equivalent syntax for the calculator used. Several sample questions in this document include screen images to illustrate the use of the statistical features of the calculator.

In these questions, it should be noted that rounding does not occur until the final answer. It should also be noted that ***z*-tables will be provided for the examination** since not all of the approved calculators contain the necessary statistical features. If tables are used for a question, the final solution may differ slightly than those provided in this document. On an examination, markers will be aware of both the table solution and the calculator solution and will not deduct marks for differences in accuracy.

A graphing calculator is essential for the Principles of Mathematics 12 Provincial Examination. The calculator must be a hand-held device designed primarily for mathematical computations involving logarithmic and trigonometric functions, for graphing functions and for performing statistical tests. Computers, calculators with a QWERTY keyboard or symbolic manipulation abilities; such as the Computer Algebraic System (CAS) and electronic writing pads will not be allowed. Students must not bring any external devices (peripherals) to support calculators such as manuals, printed or electronic cards, printers, memory expansion chips or cards, CD-ROMs, libraries or external keyboards. Students may have more than one calculator available during the examination, of which one may be a scientific calculator. Calculators may not be shared and must not have the ability to either transmit or receive electronic signals. In addition to an approved calculator, students will be allowed to use rulers, compasses, and protractors during the examination. *Calculators must not have any information programmed into the memory which would not be acceptable in paper form.* Specifically, calculators must not have any built-in notes, definitions, or libraries. There is no requirement to clear memories at the beginning of the examination but the use of calculators with built-in notes is equivalent to the use of notes in paper form. Any student deemed to have cheated on a provincial examination will receive a "0" on that examination and will be permanently disqualified from the Provincial Examination Scholarship Program.

Sample calculator-active questions and responses are provided in the Appendix: Sample Questions.

Since it is essential for students to have a graphing calculator, the following list is provided as a guideline to assist students and teachers in the selection of graphing calculators that conform to the requirements stated in the Calculator Policy on page 1. This list is taken from the approved list of graphing calculators published for the Advanced Placement Examinations, and has been adapted for the Principles of Mathematics 12 Provincial Examination. Students may still bring to the examination a scientific calculator (programmable or non-programmable) that satisfies the requirements of the Calculator Policy in addition to their graphing calculator (although one calculator is sufficient for the examination).

The following is a list of approved graphing calculators.

CASIO	HEWLETT-PACKARD	RADIO SHACK	SHARP	TEXAS INSTRUMENTS
fx-6000 series	HP 28 series	EC-4033	EL-5200	TI-81
fx-6200 series	HP 38 series	EC-4034	EL-9200 series	TI-82
fx-6300 series	HP 39G	EC-4037	EL-9300 series	TI-83
fx-6500 series			EL-9600	TI-83 Plus
fx-7000 series			EL-9900	TI-83 Plus Silver
fx 7300 series				TI-84 Plus
fx-7400 G series				TI 84 Plus Silver
fx 7500 series				TI-85
fx-7700 series				TI-86
fx-7800 series				
fx-8000 series				
fx-8500 series				
fx-8700 series				
fx-8800 series				
fx-9700 series				
fx-9750 series				
fx-1.0 series				
cfx-9800 series				
cfx-9850 series				
cfx-9850GA plus				
cfx-9950 series				
cfx-9000g				

Note: This list is not exclusive and will be updated, as necessary, to include new allowable graphing calculators.

PRINCIPLES OF MATHEMATICS 12

DESCRIPTION OF THE PROVINCIAL EXAMINATION

The Table of Specifications (page 5) outlines the curriculum organizers, sub-organizers, and the cognitive level emphases covered on the provincial examination. A detailed description of examinable material within each curriculum organizer for Principles of Mathematics will be found in the *Mathematics 10 to 12 Integrated Resource Package 2000.*

The provincial examination is divided into **two** parts:

PART A: **Multiple-choice** questions worth **73%** of the examination
(44 questions worth 1.5 marks each to give 66 marks).

PART B: **Written-response** questions worth **27%** of the examination
(5 questions worth a total of 24 marks).

It is essential that students know what quality of response is expected on written-response questions and how they are evaluated on such questions. The intent of the written-response section of the provincial examination is to test logical mathematical development of a solution.

1. Students are expected to communicate their knowledge and understanding of mathematical concepts in a clear and logical manner in all questions. The use of a graphing calculator will affect the way some of the questions are asked and the way students will be expected to answer them. In some cases, a particular method of solution will be requested; in other cases, multiple methods of solution will be encouraged. In all cases, students will be expected to show all of their work and indicate their method of solution or give justification for their written-response answers. **Full** marks will not be awarded for a correct final answer without evidence of the process used to derive that answer.

2. If, in a justification, a student refers to information produced by the graphing calculator, this information must be presented clearly in the response. For example, if a graph is used in the solution of the problem, it is important to accurately sketch the graph, showing its general shape and indicating the appropriate window dimensions.

   If the statistical features of the calculator are used, it is important to show the function with the substitution of the relevant numbers. For example: in part of the solution it is acceptable to show normalcdf (40, 40, 47, 10) or the equivalent syntax for the calculator used.

3. When using the calculator, a student should provide a decimal answer that is correct to **at least two decimal places** (unless otherwise indicated). Such rounding should occur **only** in the final step of the solution. **Exact answers** may also be requested.

A graphing calculator is essential for the Principles of Mathematics 12 Provincial Examination.
See the Calculator Policy (page 1) for a detailed description of the allowable functions of the calculator.

The time allotted for the provincial examination is **two hours**. *Students may, however, take up to 30 minutes of additional time to finish.*

PRINCIPLES OF MATHEMATICS 12

TABLE OF SPECIFICATIONS FOR THE PROVINCIAL EXAMINATION

CURRICULUM		COGNITIVE LEVEL				TOTAL
ORGANIZERS	**SUB-ORGANIZERS**	Knowledge	Understanding and Application: Lower Level	Understanding and Application: Higher Level	Higher Mental Processes	%
Problem Solving	A. Problem Solving — *Problem Solving and Cross-Topic Problems*	Integrated throughout the examination				
Patterns and Relations	B. Patterns — *Geometric Sequences and Series*	<—— 8 ——>				49
	C. Variables and Equations D. Relations and Functions — *Logarithms and Exponents*	<—— 17 ——>				
	— *Trigonometry*	<—— 24 ——>				
Shape and Space	E. 3-D Objects and 2-D Shapes — *Conics*	<—— 10 ——>				23
	F. Transformations — *Transformations*	<—— 13 ——>				
Statistics and Probability	G. Chance and Uncertainty — *Combinatorics*	<—— 8 ——>				28
	— *Probability*	<—— 10 ——>				
	— *Statistics*	<—— 10 ——>				
	TOTAL:	10	35	35	20	100

The values in this table are approximate and may fluctuate. The weighting of each topic reflects the amount of instructional time devoted to each sub-organization. Since it is not possible to test all prescribed learning outcomes, the examination will be based on a sampling of the curriculum.

Examination configuration: 66 marks in multiple-choice format (44 questions worth 1.5 marks each)
24 marks in written-response format
Total: 90 marks

DESCRIPTION OF COGNITIVE LEVELS

*The following three cognitive levels are based on a modified version of Bloom's taxonomy (*Taxonomy of Educational Objectives*, Bloom* et al.*, 1956). Bloom's taxonomy describes five cognitive domains: Knowledge, Understanding and Application, Analysis, Synthesis, and Evaluation.*

Knowledge

Knowledge questions emphasize the recognition or recall of terminology, specific facts, conventions, classifications, and notation.

Understanding and Application

Lower-Level Understanding and Application questions may require students to recognize a correct answer after performing a single computation, to substitute values into a given formula, or to identify specific characteristics of a graph or diagram.

Included at *Higher-Level Understanding and Application* are questions that may require students to form and solve equations, manipulate expressions, or produce or interpret a graph or diagram.

Higher Mental Processes

Included at this cognitive level are the processes of analysis, synthesis, and evaluation.

Analysis involves the ability to recognize unstated assumptions, to distinguish facts from hypotheses, to distinguish conclusions from statements that support them, to recognize which facts or assumptions are essential to support an argument.

Synthesis involves the production of a unique communication, the ability to make generalizations. Students may be required to apply concepts from several topics to a novel situation, or to identify and solve sub-problems that lead to a final solution.

Evaluation includes making judgements about the value of ideas, solutions, and methods. It involves the comparison of various methods of solution, the justification of results by presenting a clear argument, and the identification of logical fallacies in arguments.

Questions at the higher mental processes level subsume both *knowledge* and *understanding and application* levels.

APPENDIX:

SAMPLE QUESTIONS

A: PROBLEM SOLVING — *Problem Solving and Cross-Topic Problems*

Problem solving and cross-topic problems are integrated throughout the other strands.

B: PATTERNS AND RELATIONS (Patterns) — *Geometric Sequences and Series*

Knowledge

1. Determine the common ratio of the geometric sequence $1,\ -\frac{1}{3},\ \frac{1}{9},\ -\frac{1}{27}$.

   A. -3

   * B. $-\frac{1}{3}$

   C. $\frac{1}{3}$

   D. 3

Understanding (lower level)

2. Determine the 14th term of the geometric series: $6+12+24+\ldots$

   A. 12 288
   B. 24 576
   * C. 49 152
   D. 98 304

Understanding (higher level)

3. While training for a race, a runner increases her distance by 10% each day. If she runs 2 km on the first day, what will be her **total** distance for 26 days of training? (Accurate to 2 decimal places.)

   A. 21.67 km
   B. 23.84 km
   C. 196.69 km
   * D. 218.36 km

Higher Mental Processes

4. Evaluate: $\sum_{n=1}^{\infty} \frac{1}{3^n}$

   A. $\frac{1}{3}$

   * B. $\frac{1}{2}$

   C. $\frac{2}{3}$

   D. 1

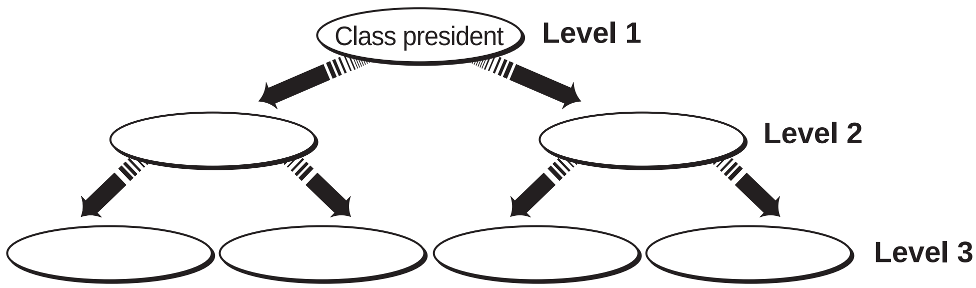
Higher Mental Processes (cross-topic)

* 5. Evaluate: $\sum_{k=3}^{5} \log_k k^2$

   A. 1
   B. 2
   * C. 6
   D. 8

Understanding (higher level)

6. A graduation class informs its members of changes in plans by telephone. The president of the class calls two members, each of whom in turn calls two other members, and so on, as shown in the diagram. By the 9th level, all members of the graduation class have been contacted. Determine how many students in total are in the graduation class.



solution

Geometric series: $1 + 2 + 4 + 8 + \ldots$

$a = 1$
$r = 2$
$n = 9$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$= \frac{1\left(1 - 2^9\right)}{1 - 2}$$

$$= 511$$

$\therefore$ there are 511 students in this graduation class.

Understanding (higher level)

7. In the World Dominoes tournament, 78 125 players are grouped 5 players at each table. One game is played by these 5 players and the winner at each table advances to the next round, and so on until the final game of 5 players. How many rounds would the ultimate winner have played (including the final round)?

solution

$$a = \frac{78\,125}{5} = 15\,625 \qquad (\text{number of tables / games in first round})$$

$$t_n = 1 \qquad r = \frac{1}{5}$$

$$n = ?$$

$$t_n = ar^{n-1}$$

$$1 = 15\,625\left(\frac{1}{5}\right)^{n-1}$$

$$\frac{1}{15\,625} = \left(\frac{1}{5}\right)^{n-1}$$

$$\left(\frac{1}{5}\right)^{6} = \left(\frac{1}{5}\right)^{n-1}$$

$$6 = n - 1$$

$$7 = n$$

∴ the winner would have played 7 rounds.

Check:

Games: 1, 5, 25, 125, 625, 3 125, 15 625
Players: 5, 25, 125, 625, 3 125, 15 625, 78 125

alternate solution

$$a = 78\,125 \qquad (\text{number of players in first round})$$

$$t_n = 5 \qquad (\text{number of players in final round})$$

$$r = \frac{1}{5}$$

$$t_n = ar^{n-1}$$

$$5 = 78\,125\left(\frac{1}{5}\right)^{n-1}$$

$$n = 7$$

∴ the winner would have played 7 rounds.

Higher Mental Processes

8. The first and second terms of a geometric sequence have a sum of 15, while the second and third terms have a sum of 60. Use an algebraic method to find the three terms.

solution

$$a + ar = 15 \qquad a(1+r) = 15$$
$$ar + ar^2 = 60 \qquad ar(1+r) = 60$$

$$\therefore \frac{ar(1+r)}{a(1+r)} = \frac{60}{15}$$
$$r = 4$$

$$a + ar = 15$$
$$a + 4a = 15$$
$$5a = 15$$
$$a = 3$$

$\therefore$ sequence is: $3,\ 12,\ 48$

C: PATTERNS AND RELATIONS (Variables and Equations)
D: PATTERNS AND RELATIONS (Relations and Functions)
— Logarithms and Exponents

Knowledge

1. The population of ABC high school is currently 1 250 students and is decreasing at an annual rate of 3%. Which expression represents the population, P, of the school 5 years from now?

   A. $P = 1\,250(1.03)^5$
   B. $P = 1\,250(1.03)^{-5}$
 * C. $P = 1\,250(0.97)^5$
   D. $P = 1\,250(0.97)^{-5}$

Knowledge

2. Change $y = 5^x$ to logarithmic form.

   A. $\log_5 x = y$
 * B. $\log_5 y = x$
   C. $\log_y 5 = x$
   D. $\log_x 5 = y$

Understanding (lower level)

3. The population of a particular country is 25 million. Assuming the population is growing continuously, the population P, in millions, t years from now can be determined by the formula $P = 25e^{0.022t}$. What will be the population, in millions, 20 years from now?

   A. 29.90
   B. 37.97
   C. 38.63
 * D. 38.82

Note: In the Principles of Mathematics 12 course, students should be familiar with using the base of e in continuous growth and decay problems.

Understanding (lower level)

* 4. Simplify: $\log_2 4^x$

A. x
* B. $2x$
C. 2^x
D. x^2

Understanding (higher level)

* 5. If $\log_4 x = a$, determine $\log_{16} x$ in terms of a.

A. $\frac{a}{4}$
* B. $\frac{a}{2}$
C. $2a$
D. $4a$

Understanding (higher level)

6. The Richter scale is used for comparing the intensities of earthquakes. On the Richter scale, each increase of 1 unit in magnitude represents a 10-fold increase in intensity as measured on a seismometer. In 1976, an earthquake in Guatemala had a magnitude of 7.5 on the Richter scale and in 1960, an earthquake in Morocco had a magnitude of 5.8. How many times as intense was the 1976 Guatemalan earthquake compared to the 1960 Moroccan earthquake?

A. 1.29
B. 1.7
C. $10^{1.29}$
* D. $10^{1.7}$

Higher Mental Processes

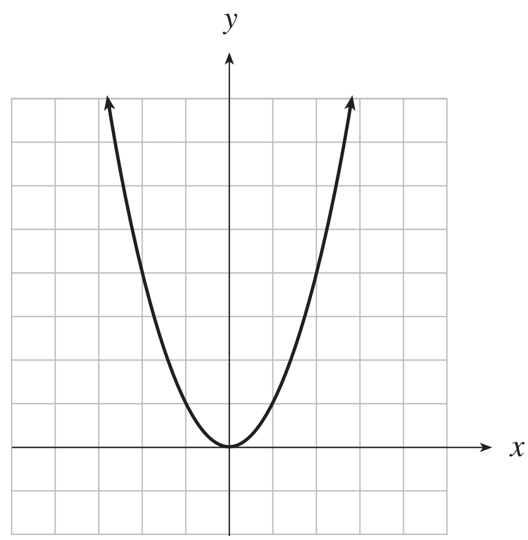
* 7. Given $\log_a 2 = x$ and $(\log_a 8)\left(a^{\log_a x}\right) = 12$, solve for a.

A. 2
B. ± 2
* C. $\sqrt{2}$
D. $\pm\sqrt{2}$

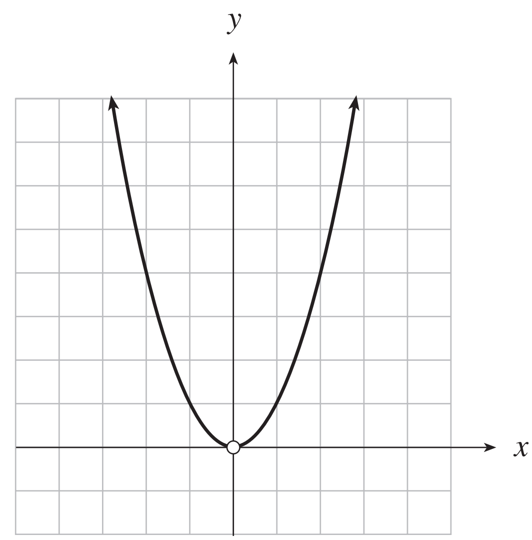
Higher Mental Processes (cross topic)

8. Which of the following is a graph of $\log_x y = 2$?

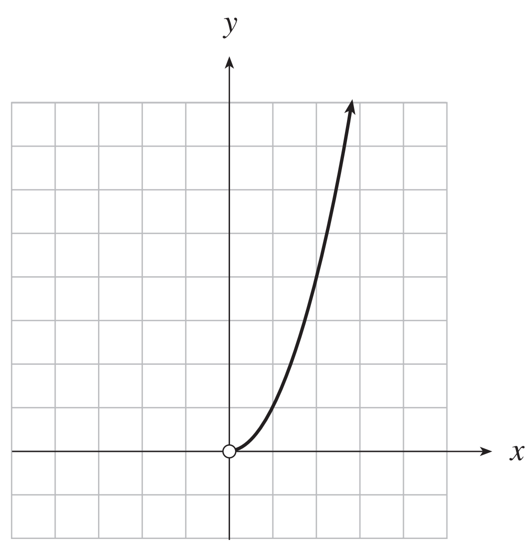
A.



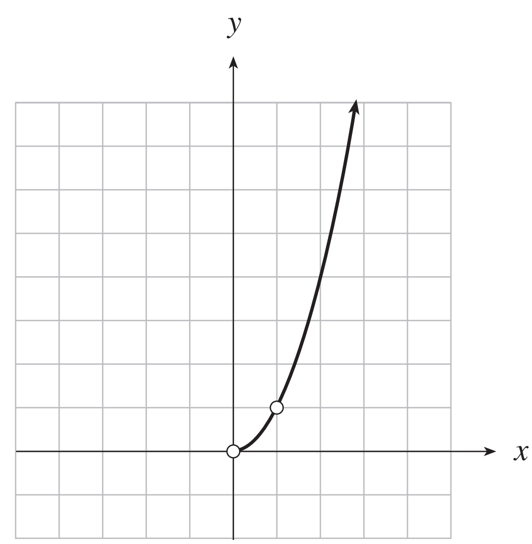
B.



C.



* D.



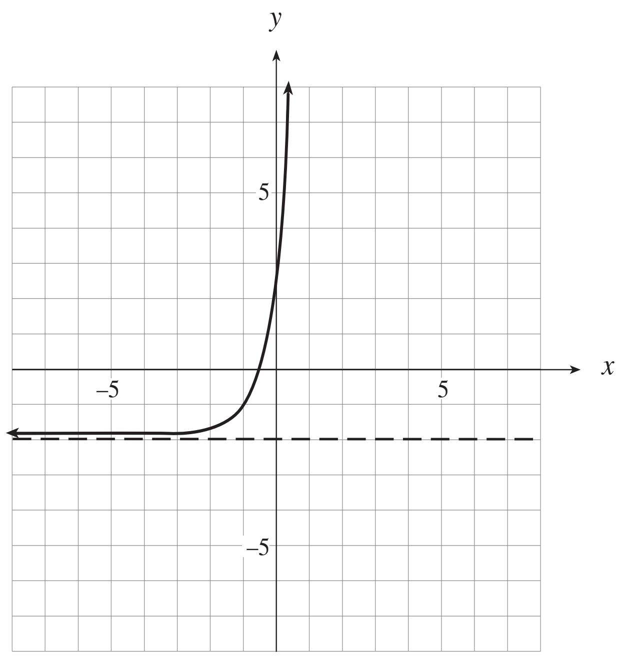
Higher Mental Processes

9. Graph $\log_5(y+2) = x+1$ on the grid below. State any asymptotes and give exact values for the x- and y-intercepts.

solution

Since $\log_5(y+2) = x+1$

$$y+2 = 5^{x+1}$$
$$y = 5^{x+1} - 2$$



asymptote: $y = -2$

x-intercept: set $y = 0$

$$0 = 5^{x+1} - 2$$
$$2 = 5^{x+1}$$
$$\log 2 = (x+1)\log 5$$
$$\frac{\log 2}{\log 5} = x+1$$

$\therefore$ x-intercept is $\dfrac{\log 2}{\log 5} - 1$

or $\log_5 2 - 1$

or x-intercept: set $y = 0$ in original equation

$$\log_5(0+2) = x+1$$
$$\log_5 2 = x+1$$
$$\therefore x = \log_5 2 - 1$$

y-intercept: set $x = 0$

$$y = 5^{0+1} - 2$$

$\therefore$ y-intercept is 3

Understanding (higher level)

10. The half-life of plutonium-239 is about 25 000 years. How many years does it take until only 36% of the plutonium still remains?

solution

$$N = C\left(\frac{1}{2}\right)^{\frac{t}{n}}$$

$$36 = 100\left(\frac{1}{2}\right)^{\frac{t}{25\,000}}$$

$$0.36 = \left(\frac{1}{2}\right)^{\frac{t}{25\,000}}$$

$$\log 0.36 = \frac{t}{25\,000}\log\frac{1}{2}$$

$$t = \frac{25\,000\log 0.36}{\log\frac{1}{2}}$$

$$t \approx 36\,848 \text{ years}$$

Higher Mental Processes

11. It is estimated that 20% of a certain radioactive substance decays in 30 hours. What is the half-life of the substance?

solution

If 20% of the substance decays, then 80% remains.

$$N = C\left(\frac{1}{2}\right)^{\frac{t}{n}}$$

$$80 = 100\left(\frac{1}{2}\right)^{\frac{30}{n}}$$

$$0.80 = \left(\frac{1}{2}\right)^{\frac{30}{n}}$$

$$\log 0.80 = \frac{30}{n}\log\frac{1}{2}$$

$$n = \frac{30\log\frac{1}{2}}{\log 0.80}$$

$$n \approx 93.19 \text{ hours}$$

Note: The half-life is the time it takes until a substance decays to half its original amount.

Higher Mental Processes

12. The population of Canada is 30 million people and is growing at an annual rate of 1.4%.
The population of Germany is 80 million people and is decreasing at an annual rate of 1.7%.
In how many years will the population of Canada be equal to the population of Germany?
(Use logarithms to solve the resulting equation and answer accurate to two decimal places.)

solution

$$30(1+0.014)^n = 80(1-0.017)^n$$

$$30(1.014)^n = 80(0.983)^n$$

$$\log 30 + n\log(1.014) = \log 80 + n\log(0.983)$$

$$n\big(\log(1.014) - \log(0.983)\big) = \log 80 - \log 30$$

$$n = \frac{\log 80 - \log 30}{\log(1.014) - \log(0.983)}$$

$$n = 31.59 \text{ years}$$

alternate solution

$$30(1+0.014)^n = 80(1-0.017)^n$$

$$30(1.014)^n = 80(0.938)^n$$

$$\left(\frac{1.014}{0.983}\right)^n = \frac{8}{3}$$

$$n\log\left(\frac{1.014}{0.983}\right) = \log\frac{8}{3}$$

$$n = \frac{\log\frac{8}{3}}{\log\left(\frac{1.014}{0.983}\right)}$$

$$n = 31.59 \text{ years}$$

C: PATTERNS AND RELATIONS (Variables and Equations)
D: PATTERNS AND RELATIONS (Relations and Functions) — *Trigonometry*

Knowledge

* 1. The point $(m,\ n)$ is the point of intersection of the terminal arm of angle θ in standard position and the unit circle centered at $(0,\ 0)$. Determine the value of $\sin\theta$.

A. m

* B. n

C. $\dfrac{m+n}{2}$

D. $\dfrac{m-n}{2}$

Understanding (lower level)

* 2. Determine the period of $y = \sin\dfrac{2\pi}{3}(x-6)$.

* A. 3
B. 6
C. 3π
D. 4π

Understanding (lower level)

3. Convert 183° to radians.

* A. 3.19
B. 6.33
C. 0.32
D. 2.58

Understanding (lower level)

* 4. Solve: $1 + 2\sin x = 0$, $0 \le x < 2\pi$ Give exact solution(s).

A. $-\dfrac{\pi}{6}$

B. $\dfrac{2\pi}{3},\ \dfrac{5\pi}{3}$

C. $\dfrac{2\pi}{3},\ -\dfrac{\pi}{3}$

* D. $\dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$

Understanding (higher level)

* 5. Determine the range of the function $y = 6\cos\frac{1}{2}(x-3)+4$.

A. $-6 \le y \le 6$
B. $1 \le y \le 7$
C. $-4 \le y \le 4$
* D. $-2 \le y \le 10$

Understanding (higher level)

* 6. Which of the following lines is an asymptote for the graph of $y = \csc x$?

A. $x = 1$

B. $x = \frac{\pi}{2}$

C. $x = \frac{3\pi}{4}$

* D. $x = \pi$

Understanding (higher level)

7. At a seaport, the depth of the water, d, in metres, at time t hours, during a certain day is given by:

$$d = 3.4\sin 2\pi\frac{(t-7.00)}{10.6}+2.8$$

On that day, determine the depth of the water at 6:30 p.m.

A. 1.39 m
B. 1.81 m
C. 2.80 m
* D. 4.53 m

Note: Students must express time in decimals of hours on a 24-hour clock in problems such as #7.

Higher Mental Processes

* 8. Determine the domain of $f(x) = \tan 2x$.

A. all real numbers

* B. all real numbers,
$x \neq (2n+1)\frac{\pi}{4}$, ($n$ is any integer)

C. all real numbers,
$x \neq (2n+1)\frac{\pi}{2}$, ($n$ is any integer)

D. all real numbers,
$x \neq (2n+1)\pi$, (n is any integer)

Higher Mental Processes

* 9. Point A$(-a,\ b)$ is in quadrant II and lies on the terminal arm of angle θ in standard position. Point B is the point of intersection of the terminal arm of θ and the unit circle centered at $(0,\ 0)$. Determine the x-coordinate of B in terms of a and b.

* A. $\frac{-a}{\sqrt{a^2+b^2}}$

B. $\frac{-b}{\sqrt{a^2+b^2}}$

C. $\frac{a}{\sqrt{a^2+b^2}}$

D. $\frac{b}{\sqrt{a^2+b^2}}$

Understanding (higher level)

10. A Ferris wheel has a diameter of 60 m and its centre is 32 m above the ground. It rotates once every 48 seconds. Jack gets on the Ferris wheel at its lowest point and then the wheel starts to rotate.

    a) Determine a sinusoidal equation that gives Jack's height, h, above the ground as a function of the elapsed time, t, where h is in metres and t is in seconds.

    b) Determine the time, t, when Jack will be 38 m above the ground in the first rotation of the Ferris wheel.

solution

Answers can vary for both a) and b). Some answers are:

a)

$$h = -30\cos\frac{\pi}{24}t + 32$$

$$h = 30\cos\frac{\pi}{24}(t - 24) + 32$$

$$h = 30\sin\frac{\pi}{24}(t - 12) + 32$$

b)

$$38 = -30\cos\frac{\pi}{24}t + 32$$

$$t = 13.54 \text{ seconds}$$

Note: Students could obtain this solution by finding the smallest positive zero of the function $Y_1 = 38 + 30\cos\frac{\pi}{24}t - 32$.

Understanding (higher level)

11. Solve $\cos^2 x = \cos x$ over the set of real numbers. Give exact value solutions.

solution

$$\cos^2 x - \cos x = 0$$

$$\cos x(\cos x - 1) = 0$$

$$\cos x = 0 \qquad\qquad \cos x - 1 = 0$$

$$\cos x = 1$$

$$x = \frac{\pi}{2},\ \frac{3\pi}{2} \qquad\qquad x = 0$$

$\therefore$ general solution $x = \frac{\pi}{2} + 2n\pi \qquad x = 0 + 2n\pi$

$$x = \frac{3\pi}{2} + 2n\pi \qquad\qquad x = 2n\pi$$

or

$$x = \frac{\pi}{2} + n\pi \qquad (n \text{ is any integer})$$

Note: When asked to solve a trigonometric equation "over the set of real numbers", it is expected that students should use radian measure.

Higher Mental Processes

12. Prove the following identity: $\dfrac{\sin 2x}{1-\cos 2x} = \cot x$

solution

LEFT SIDE	RIGHT SIDE
$\dfrac{\sin 2x}{1-\cos 2x}$ $= \dfrac{2\sin x\cos x}{1-\left(1-2\sin^2 x\right)}$ $= \dfrac{2\sin x\cos x}{2\sin^2 x}$ $= \dfrac{\cos x}{\sin x}$ $= \cot x$	$\cot x$

LS = RS

∴ since the left side = the right side, the identity is proved algebraically

Note: It should be noted that a numerical or graphical justification of an identity does not prove the identity.

E: SHAPE AND SPACE (3-D Objects and 2-D Shapes) — *Conics*

Knowledge

* 1. Determine the vertex of $x = -2(y-3)^2 - 5$.

- * A. $(-5,\ 3)$
- B. $(-3,\ 5)$
- C. $(3,\ 5)$
- D. $(5,\ 3)$

Knowledge

* 2. Identify the conic that is described by $x^2 + 6y^2 - 18y - 45 = 0$.

- A. circle
- * B. ellipse
- C. parabola
- D. hyperbola

Knowledge

* 3. Which conic is generated by the intersection of a plane and a double-napped cone when the intersecting plane is parallel to the generator of the cone?

- A. circle
- B. ellipse
- * C. parabola
- D. hyperbola

Understanding (lower level)

* 4. Determine the vertices of $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

A. $(-4,\ 0), (4,\ 0)$
B. $(-2,\ 0), (2,\ 0)$
* C. $(0,\ 3), (0,\ -3)$
D. $(0,\ 9), (0,\ -9)$

Note: It should be noted that the vertices of an ellipse are defined to be the coordinates of the endpoints of the major axis.

Understanding (higher level)

* 5. Determine the slopes of the asymptotes of $\frac{(x-2)^2}{36} - \frac{(y+3)^2}{9} = 1$.

A. $\pm\frac{1}{4}$
* B. $\pm\frac{1}{2}$
C. ± 2
D. ± 4

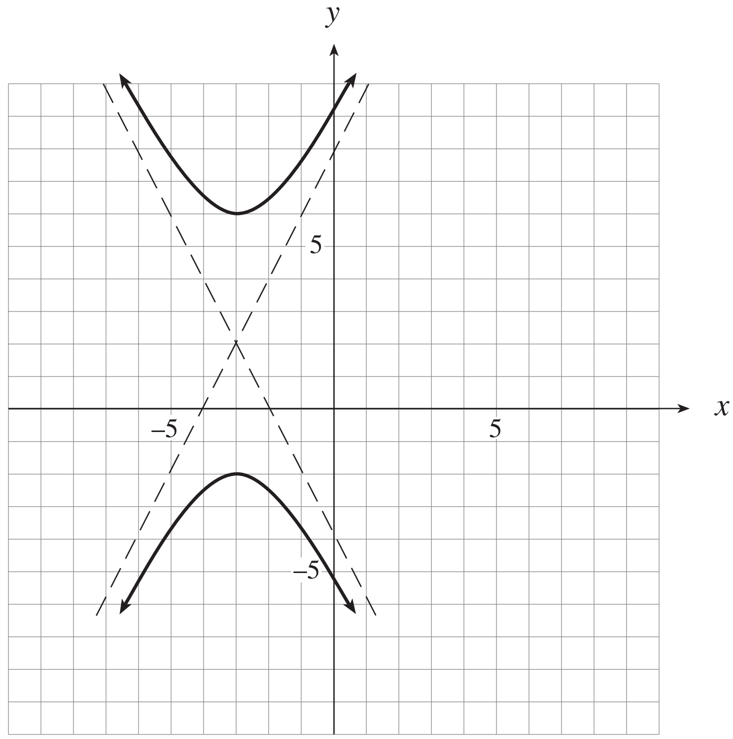
Understanding (higher level)

* 6. If m and n are positive integers, determine the radius of the circle $mx^2 + my^2 - n = 0$.

* A. $\sqrt{\frac{n}{m}}$
B. $\sqrt{\frac{m}{n}}$
C. $\frac{n}{m}$
D. $\frac{m}{n}$

Understanding (higher level)

* 7. Determine the standard form equation of the conic graphed below.



A. $\frac{(x+3)^2}{4} - \frac{(y-2)^2}{16} = 1$

* B. $\frac{(x+3)^2}{4} - \frac{(y-2)^2}{16} = -1$

C. $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{16} = 1$

D. $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{16} = -1$

Higher Mental Processes

* 8. Give conditions for the constants A, C and D such that the following equation is a parabola with a horizontal axis of symmetry:

$$Ax^2 + Cy^2 + Dx + y = 0$$

A. $A \neq 0$, $C = 0$, $D = 0$
B. $A \neq 0$, $C = 0$, $D \neq 0$
* C. $A = 0$, $C \neq 0$, $D \neq 0$
D. $A = 0$, $C \neq 0$, $D = 0$

Higher Mental Processes

* 9. If $AC < 0$, which conic could be represented by the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$?

A. circle
B. ellipse
C. parabola
* D. hyperbola

Understanding (higher level)

10. Given the conic $25x^2 + 16y^2 + 100x - 32y - 284 = 0$:

    a) change the equation to standard form.

    b) graph the conic on the grid below.

solution

a)

$$25\left(x^2 + 4x\right) + 16\left(y^2 - 2y\right) = 284$$

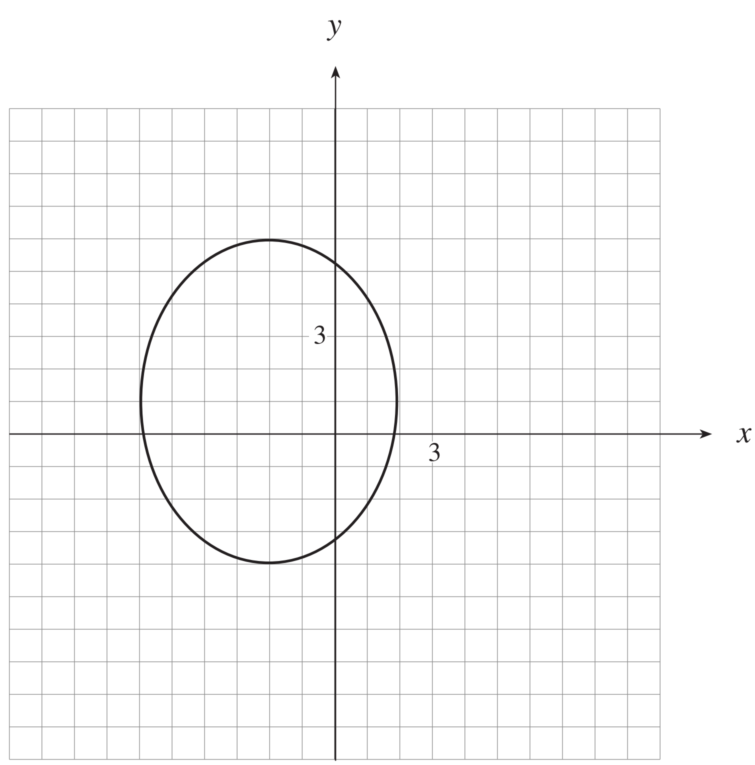
$$25\left(x^2 + 4x + 4\right) + 16\left(y^2 - 2y + 1\right) = 284 + 100 + 16$$

$$25(x+2)^2 + 16(y-1)^2 = 400$$

$$\frac{25(x+2)^2}{400} + \frac{16(y-1)^2}{400} = \frac{400}{400}$$

$$\frac{(x+2)^2}{16} + \frac{(y-1)^2}{25} = 1$$

b)



F: SHAPE AND SPACE (Transformations) — *Transformations*

Knowledge

1. If the graph of $y = f(x)$ is translated 5 units to the left, determine the resulting equation.

   A. $y - 5 = f(x)$
   B. $y + 5 = f(x)$
   C. $y = f(x - 5)$
   * D. $y = f(x + 5)$

Knowledge

2. What is the inverse of the relation $y = x^3$?

   A. $y = \frac{1}{x^3}$

   * B. $x = y^3$

   C. $y = (-x)^3$

   D. $x = y^{\frac{1}{3}}$

Knowledge

* 3. What happens to the graph of $y = x^2$ if the equation is changed to $y = x^2 - 2$?

   A. The graph is translated 2 units to the right.
   B. The graph is translated 2 units to the left.
   C. The graph is translated 2 units up.
   * D. The graph is translated 2 units down.

Understanding (lower level)

* 4. How is the graph $5y = \sqrt{x}$ related to the graph $y = \sqrt{x}$?

A. $y = \sqrt{x}$ has been vertically translated 5 units up.

B. $y = \sqrt{x}$ has been expanded vertically by a factor of 5.

* C. $y = \sqrt{x}$ has been compressed vertically by a factor of $\frac{1}{5}$.

D. $y = \sqrt{x}$ has been compressed horizontally by a factor of $\frac{1}{5}$.

Understanding (lower level)

* 5. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in

A. the y-axis.
* B. the x-axis.
C. the line $y = x$.
D. the line $y = -x$.

Understanding (lower level)

6. Simplify: $f^{-1}(f(x))$

* A. x

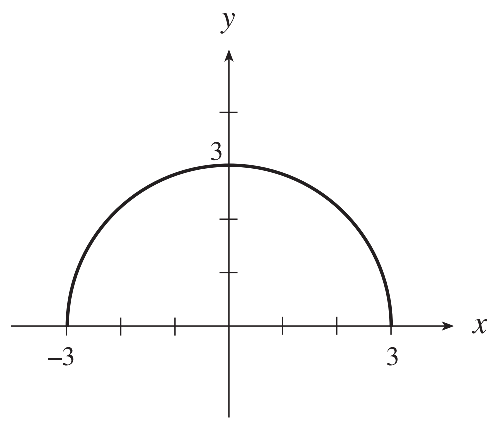
B. $-x$

C. $\frac{1}{x}$

D. $-\frac{1}{x}$

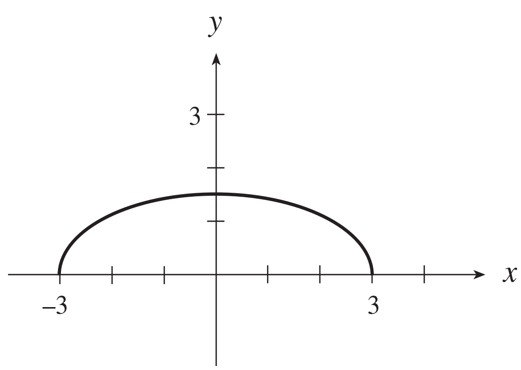
Understanding (lower level)

***** 7. The graph of $y = \sqrt{9 - x^2}$ is shown below.

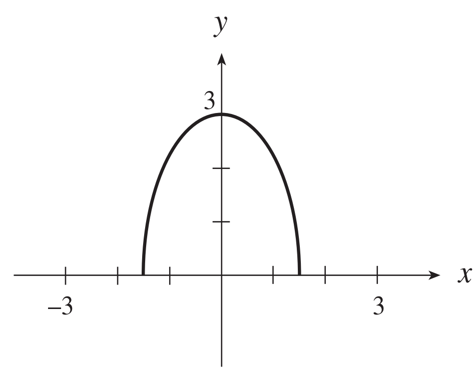


Which of the following graphs represents $2y = \sqrt{9 - x^2}$?

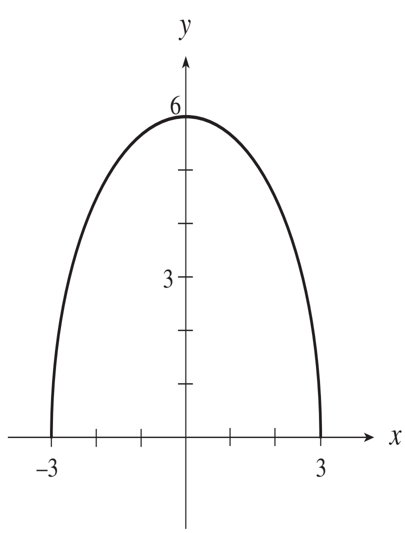
* A.



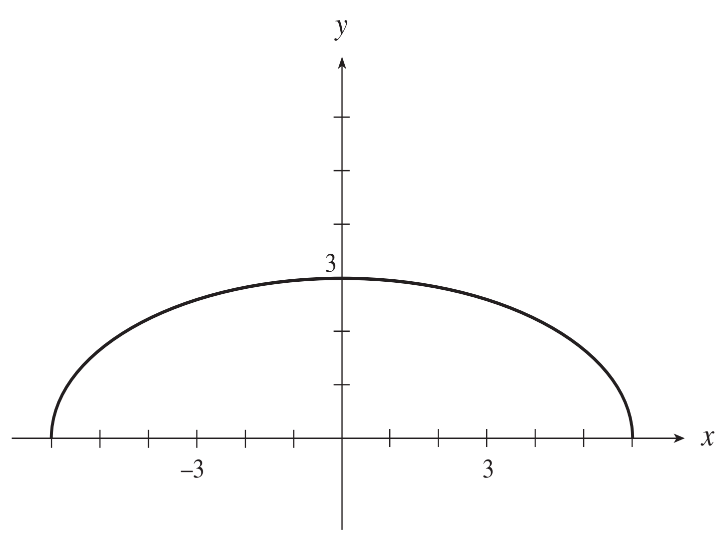
B.



C.

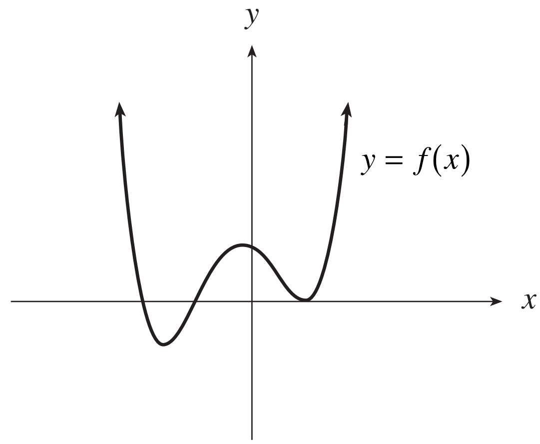


D.



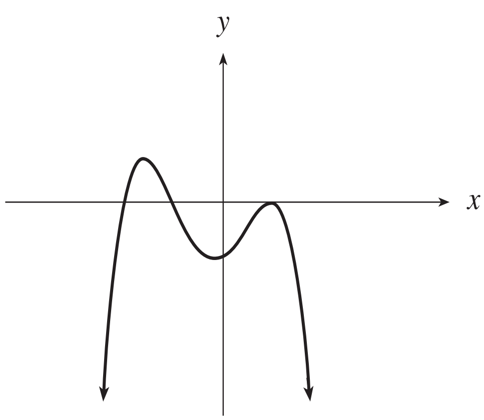
Understanding (higher level)

* 8. The graph of the function $y = f(x)$ is shown below.

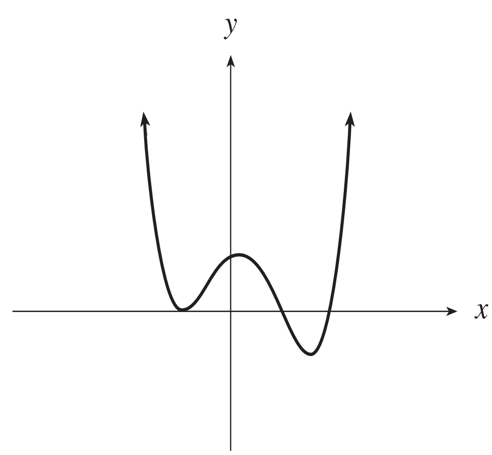


Which of the following is a graph of $y = |f(x)|$?

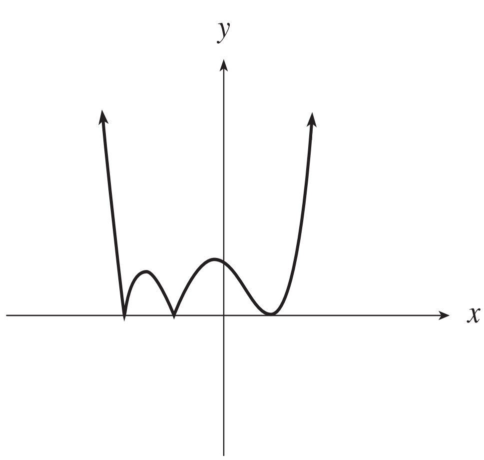
A.



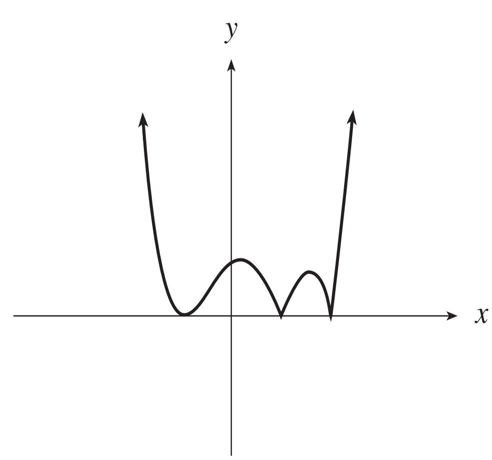
B.



* C.



D.



Understanding (higher level)

* 9. Given the function $f(x) = (x-1)^3 + 2$, determine $f^{-1}(x)$, the inverse function.

A. $f^{-1}(x) = \sqrt[3]{x+2} + 1$

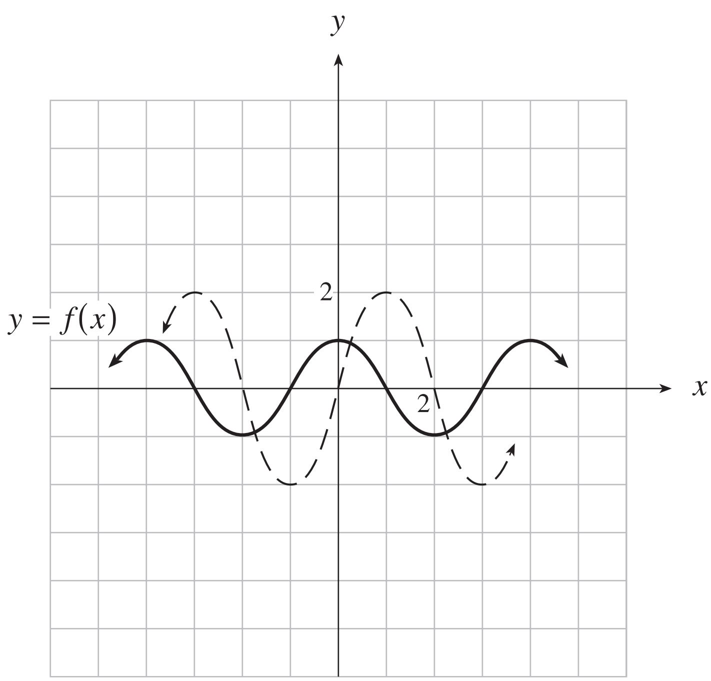
* B. $f^{-1}(x) = \sqrt[3]{x-2} + 1$

C. $f^{-1}(x) = \sqrt[3]{x+2} - 1$

D. $f^{-1}(x) = \sqrt[3]{x-2} - 1$

Understanding (higher level)

10. In the diagram below, $y = f(x)$ is graphed as a solid line.



Which equation is defined by the broken line?

A. $y = 2f(x+1)$

B. $y = f(2x-1)$

C. $y = f(2x+1)$

* D. $y = 2f(x-1)$

Higher Mental Processes

11. The function $y = f(x)$ is transformed to $y = f(2x + 4)$. Identify the horizontal expansion or compression factor, then the translation to the graph of the function.

   A. horizontal expansion by a factor of 2, then a translation of 4 units left.

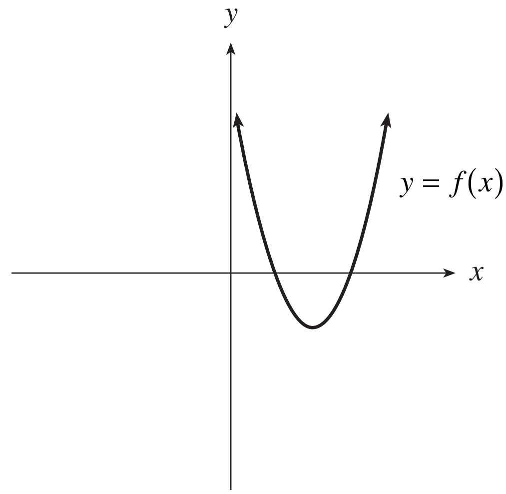
   B. horizontal compression by a factor of $\frac{1}{2}$, then a translation of 4 units left.

   C. horizontal expansion by a factor of 2, then a translation of 2 units left.

 * D. horizontal compression by a factor of $\frac{1}{2}$, then a translation of 2 units left.

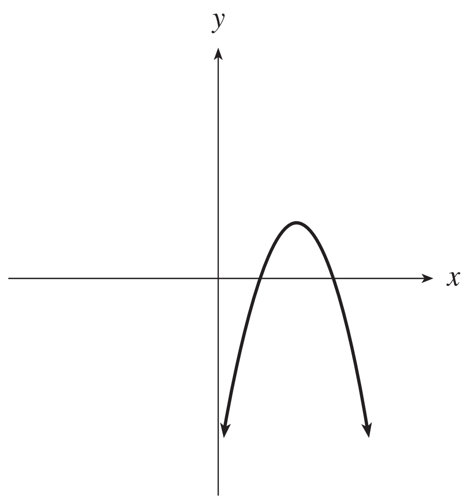
Higher Mental Processes

12. The graph of the function $y = f(x)$ is shown below.

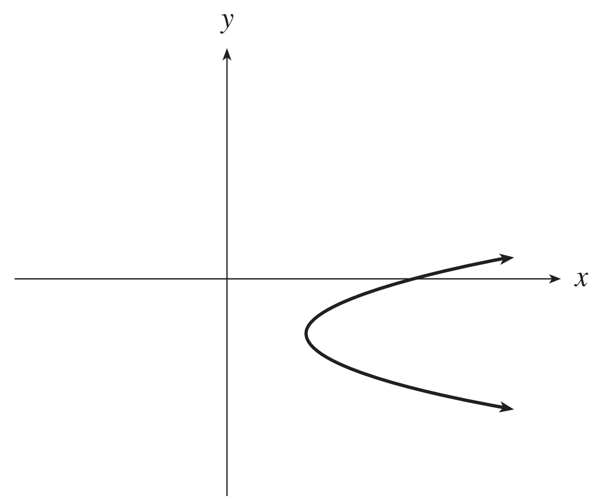


Which of the following is a graph of $y = \dfrac{1}{f(x)}$?

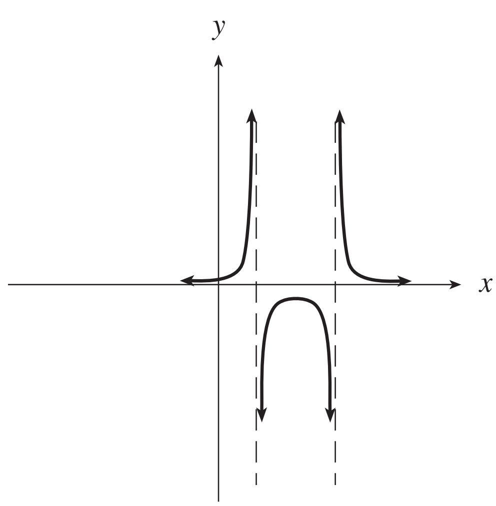
A.



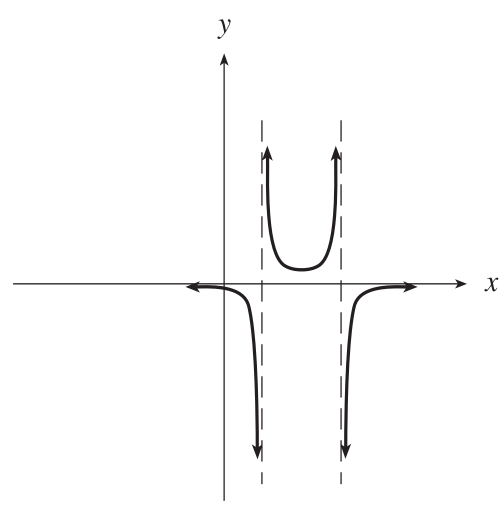
B.



* C.

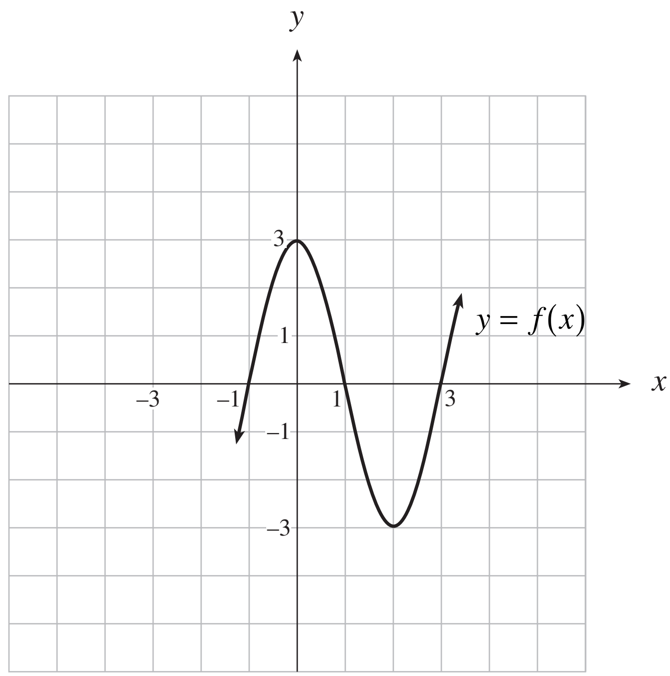


D.

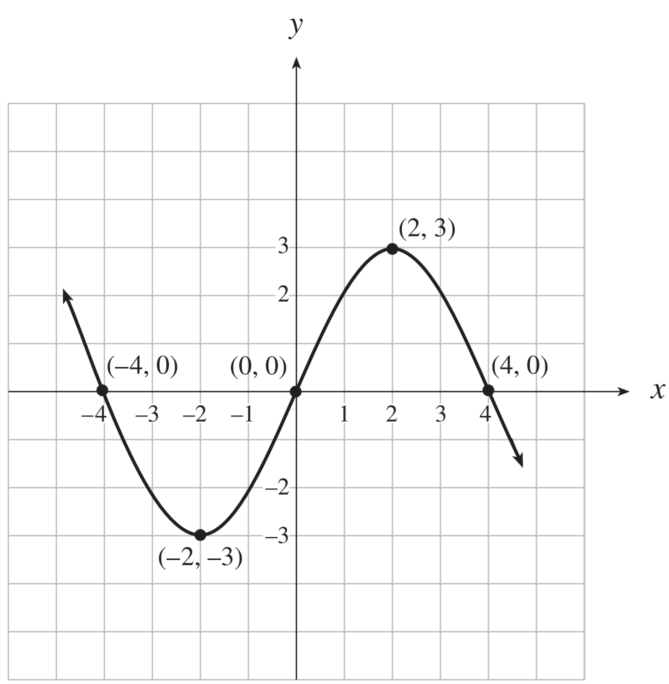


Understanding (higher level)

13. The graph of $y = f(x)$ is shown below. On the grid provided, sketch the graph of $y = -f\left(\frac{1}{2}(x+2)\right)$. Give coordinates for three points on your graph.

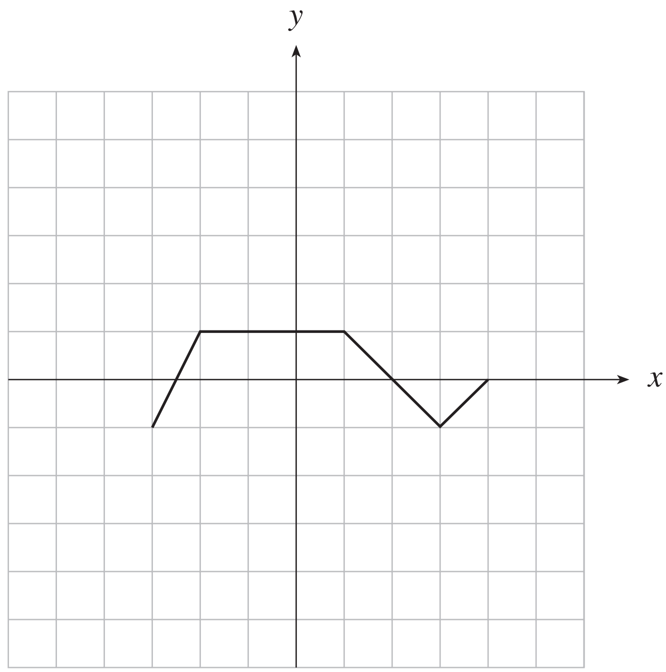


solution

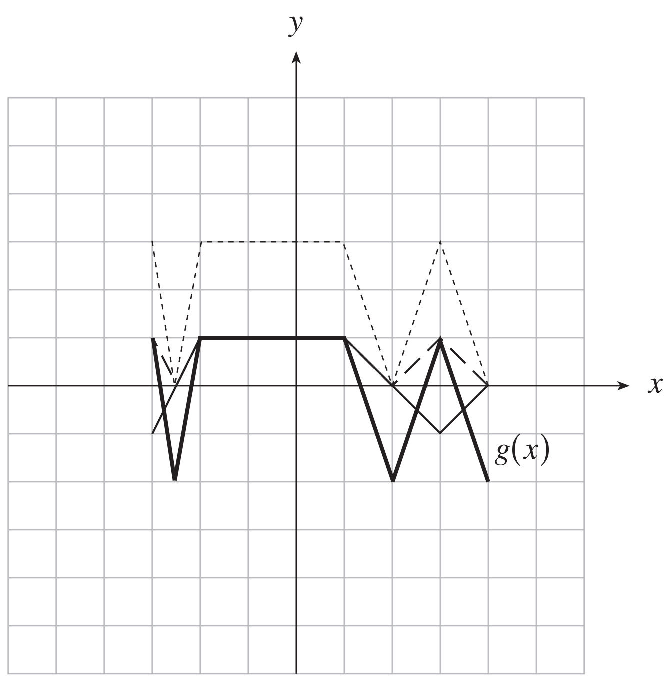


Higher Mental Processes

14. Given the graph of $f(x)$ below, sketch $g(x) = 3|f(x)| - 2$.



solution

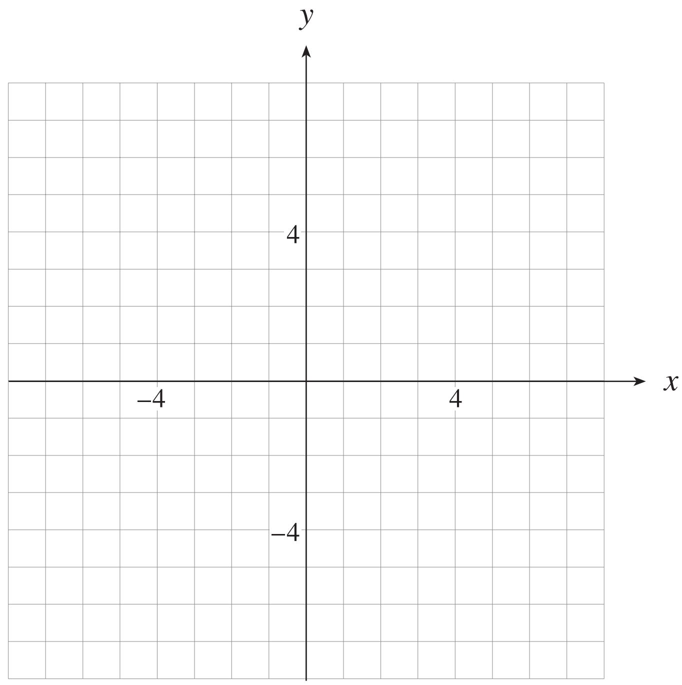


Higher Mental Processes

15. For the function $f(x) = \dfrac{1}{x+3}$:

    a) determine the equation that defines the inverse function, $f^{-1}(x)$.

    b) sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the grid provided.



solution

a)

$$y = \frac{1}{x+3}$$

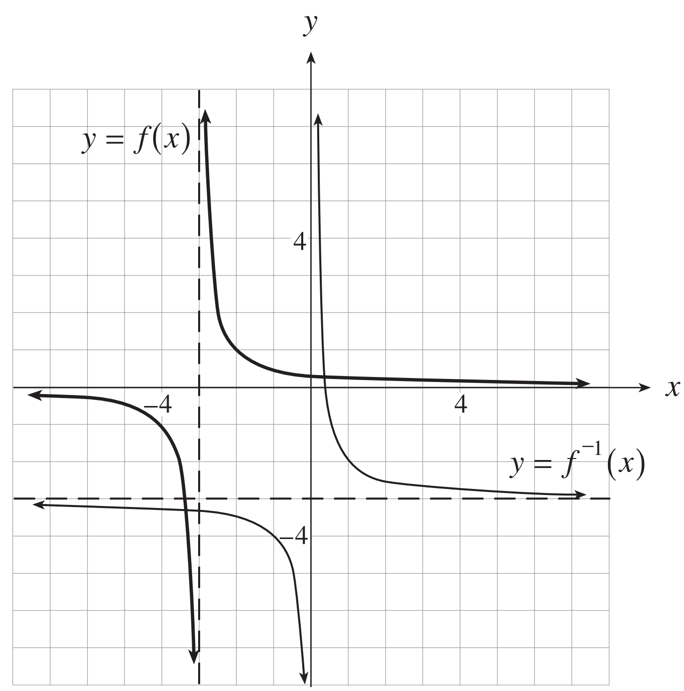
$$\text{Inverse: } x = \frac{1}{y+3}$$

$$y + 3 = \frac{1}{x}$$

$$y = \frac{1}{x} - 3$$

$$f^{-1}(x) = \frac{1}{x} - 3$$

b)



G: STATISTICS AND PROBABILITY (Chance and Uncertainty) — *Combinatorics*

Knowledge

1. How many terms are in the expansion $\left(2x - \frac{1}{y}\right)^{10}$?

   A. 9
   B. 10
   * C. 11
   D. 12

Knowledge

2. A bowl contains an apple, a pear, a plum, and a banana. How many different pairs of fruit can be selected from the bowl?

   A. ${}_4P_2$
   B. ${}_2P_4$
   * C. ${}_4C_2$
   D. ${}_2C_4$

Knowledge

3. A special combination lock that has 60 numbers on the dial works by turning it first to the right, then to the left, and then to the right, with 3 different selected numbers needed to open the lock. The selection of these 3 numbers is an example of

   * A. a permutation.
   B. a combination.
   C. both a combination and a permutation.
   D. neither a combination nor a permutation.

Understanding (lower level)

4. There are 45 multiple-choice questions on an exam with 4 possible answers for each question. How many different ways are there to complete the test?

   A. 45
   B. 148 995
   C. 3 575 880
   * D. 4^{45}

Understanding (lower level)

5. A breakfast special consists of choosing one item from each category in the following menu.

Juice:	apple, orange, grapefruit
Toast:	white, brown
Eggs:	scrambled, fried, poached
Beverage:	coffee, tea, milk

How many different breakfast specials are possible?

A. 11
B. 48
* C. 54
D. 96

Understanding (lower level)

6. North American area codes are three digit numbers. Before 1995, area codes had the following restrictions: the first digit could not be 0 or 8, the second digit was either 0 or 1, and the third digit was any number from 1 through 9 inclusive. Under these rules, how many different area codes were possible?

A. 112
B. 120
* C. 144
D. 504

Understanding (lower level)

7. Katie wants to colour a rainbow. She knows the seven colours that make up a rainbow, but can't remember the correct order. How many different ways could the colours be arranged assuming each colour is used only once?

A. 28
B. 128
C. 720
* D. 5 040

Understanding (higher level)

* 8. Simplify the following expression without using the factorial symbol $\dfrac{(n-2)!(n+1)!}{(n!)^2}$.

A. $\dfrac{1}{n}$

B. $\dfrac{1}{n-1}$

C. $\dfrac{n-1}{n(n+1)}$

* D. $\dfrac{n+1}{n(n-1)}$

Understanding (higher level)

9. The 10th term of the expansion of $\left(x-\dfrac{1}{2}\right)^n$ is $-\dfrac{1\,001}{256}x^5$. Determine n.

A. 13
* B. 14
C. 15
D. not possible to determine n from the given information

Understanding (higher level)

10. Linda and Sam play a tennis match. The first person to win 2 games wins the match. In how many different ways can a winner be determined?

A. 3
B. 5
* C. 6
D. 8

Understanding (higher level)

11. How many 6 digit numbers greater than 800 000 can be made from the digits 1, 1, 5, 5, 5, 8 ?

* A. 10
B. 60
C. 64
D. 120

Understanding (higher level)

12. In how many ways can four colas, three iced teas, and three orange juices be distributed among ten graduates if each graduate is to receive one beverage?

  A. 36
* B. 4 200
  C. 604 800
  D. 3 628 800

Higher Mental Processes

*13. Solve for n: ${}_nP_2 = 42$

  A. 2
  B. 6
* C. 7
  D. 42

Higher Mental Processes

14. Assuming that at least one coin is used, how many different sums of money can be made from the following coins: a penny, a nickel, a dime, a quarter, and a dollar?

  A. 16
* B. 31
  C. 32
  D. 120

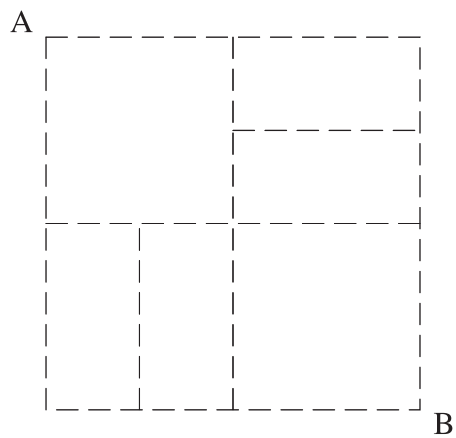
Higher Mental Processes

15. Which term in the expansion of $\left(\frac{1}{2x^2} - x^3\right)^{10}$ is a constant?

  A. 4^{th}
* B. 5^{th}
  C. 6^{th}
  D. 11^{th}

Higher Mental Processes

16. Moving only to the right or down, how many different routes exist to get from point A to point B?



A. 5
B. 6
C. 7
* D. 8

Understanding (higher level)

17. Sears wants to build 8 new stores in western Canada. They have the following information.

Province	Number of stores to be built	Number of possible locations
BC	2	6
Alberta	3	5
Saskatchewan	1	4
Manitoba	2	5

If Sears wants to study all possibilities for the location of the 8 new stores, how many different possibilities would the company have to consider?

solution

Number of possible combinations for each province:

BC = ${}_6C_2$

Alberta = ${}_5C_3$

Saskatchewan = ${}_4C_1$

Manitoba = ${}_5C_2$

∴ total number of possibilities =

BC × Alberta × Sask. × Man. =

$$\left({}_6C_2\right)\left({}_5C_3\right)\left({}_4C_1\right)\left({}_5C_2\right) = 6\ 000$$

∴ the company would have to consider 6 000 possibilities

Understanding (higher level)

18. What is the 10^{th} term of $\left(2x - \frac{1}{y}\right)^{10}$?

solution

$(k+1)^{\text{th}}$ term: $t_{k+1} = {}_nC_k\, a^{n-k}\, b^k$

$\therefore$ for the 10^{th} term, $k = 9$

$$\begin{aligned} t_{10} &= {}_{10}C_9 (2x)^{10-9}\left(-\frac{1}{y}\right)^9 \\ &= -(10)(2x)\left(\frac{1}{y^9}\right) \\ &= -\frac{20x}{y^9} \end{aligned}$$

Note: When using the $(k+1)^{\text{th}}$ term formula for the expansion of $(a+b)^n$, we will always assume that the powers of a are decreasing.

Understanding (higher level)

19. Solve: $\dfrac{n!}{(n-2)!3!} = 5$

solution

$$\begin{aligned} \frac{n(n-1)(n-2)!}{(n-2)!\,3!} &= 5 \\ \frac{n(n-1)}{6} &= 5 \\ n(n-1) &= 30 \\ n^2 - n - 30 &= 0 \\ (n-6)(n+5) &= 0 \end{aligned}$$

$n = -5 \qquad n = 6$

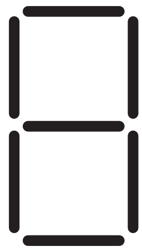
$\Downarrow$

reject

$\therefore\ n = 6$

Higher Mental Processes

20. Numbers are formed on a calculator using seven lines which are either lit or not lit. The diagram below shows the number 8 formed using all 7 lines lit. How many different symbols can be created by lighting one or more of these 7 lines? (Count all the symbols, not just the ones that represent numbers.)



solution

$$_7C_1 + {_7C_2} + {_7C_3} + {_7C_4} + {_7C_5} + {_7C_6} + {_7C_7}$$

$$= 7 + 21 + 35 + 35 + 21 + 7 + 1$$

$= 127$ ways

alternate solution 1

Observe from Pascal's triangle:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 | 7 21 35 35 21 7 1 |

Sum of these numbers = 127 ways

alternate solution 2

$$_nC_0 + {_nC_1} + {_nC_2} + \ldots + {_nC_n} = 2^n$$

$\therefore$ if $n = 7 \Rightarrow$ sum $= 2^7$

$\therefore$ sum of $_7C_1 + {_7C_2} + \ldots + {_7C_7} = 2^7 - 1$

$= 127$ ways

Higher Mental Processes

21. There are five boys and six girls on a grad committee.

    a) In how many ways can a sub-committee of two boys and two girls be selected from the committee?

    b) In how many ways can a sub-committee of four people be selected if there must be **at least** one girl on the sub-committee?

solution

a)

number of ways two boys can be selected $= {}_5C_2$
number of ways two girls can be selected $= {}_6C_2$

$\therefore$ number of possible ways for a sub-committee of two boys and two girls to be selected

$= \left({}_5C_2\right)\left({}_6C_2\right) = (10)(15) = 150$ ways

b)

If there is at least one girl on the sub-committee, then we must consider the following cases:

1 girl, 3 boys $= \left({}_6C_1\right)\left({}_5C_3\right) = (6)(10) = 60$ ways

2 girls, 2 boys $= \left({}_5C_2\right)\left({}_6C_2\right) = (10)(15) = 150$ ways

3 girls, 1 boy $= \left({}_6C_3\right)\left({}_5C_1\right) = (20)(5) = 100$ ways

4 girls, 0 boys $= \left({}_6C_4\right)\left({}_5C_0\right) = (15)(1) = 15$ ways

$\therefore$ there are $60 + 150 + 100 + 15 = 325$ different ways in which a sub-committee of four students with at least one girl can be selected.

alternate solution

b)

Consider how many ways there could be a sub-committee of four students with **no** girls:

$$\left({}_6C_0\right)\left({}_5C_4\right) = 5$$

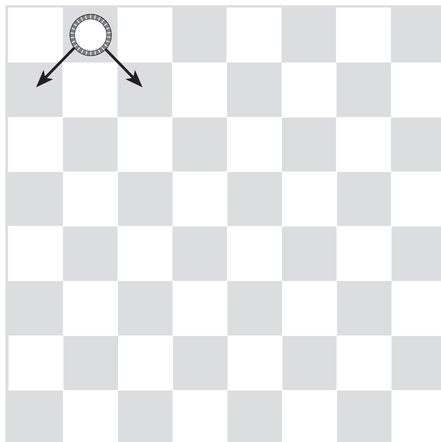
Total number of ways for a sub-committee of four students to be selected from eleven students

$$= {}_{11}C_4 = 330$$

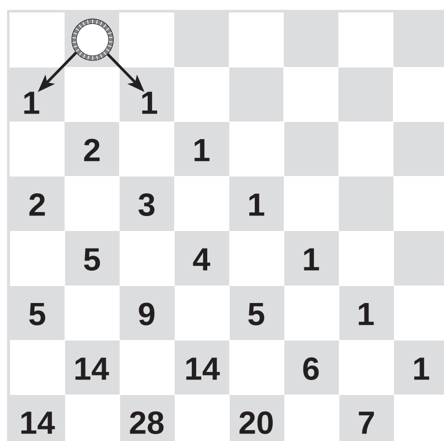
$\therefore$ number of ways in which a sub-committee has **at least** one girl $= 330 - 5 = 325$

Higher Mental Processes

22. A checkerboard is an 8×8 game board, as shown below. Game pieces can travel only diagonally on the dark squares, one diagonal square at a time, and only in a downward direction. If a checker is placed as shown, how many possible paths are there for the checker to reach the opposite side of the game board?



solution



Therefore, the total number of possible paths for the checker to reach the other side is the sum of the number of possible paths to each of the bottom squares:

$14 + 28 + 20 + 7 = 69$ possible paths

G: STATISTICS AND PROBABILITY (Chance and Uncertainty) — *Probability*

Knowledge

1. For each probability experiment, the set of all possible outcomes is called

   A. odds.
   B. event.
   C. element.
   * D. sample space.

Knowledge

2. Which of the following pair of events is dependent?

   * A. Two cards are selected from a well-shuffled deck of cards and the experiment is carried out without replacement. The first event is drawing a jack. The second event is drawing another jack.

   B. Two cards are selected from a well-shuffled deck of cards and the experiment is carried out with replacement. The first event is drawing an ace of hearts. The second event is drawing a black 5.

   C. A fair die is rolled and a fair coin is tossed. The first event is rolling an odd number on the die. The second event is obtaining a tail on a flip of the coin.

   D. A fair coin is tossed twice. The first event is obtaining a head on the first flip of the coin. The second event is obtaining a head on the second flip of the coin.

Understanding (lower level)

3. Two fair dice are rolled. A sample space is provided below.

Second Die

First Die

	1	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Consider the following events:

A: The sum of the two dice is 5.
B: The first die rolled is a 1.
C: The second die rolled is a 4.
D: The product of the two dice is 6.

Which of the two events given above are mutually exclusive?

A. A and B
B. A and C
C. B and C
* D. C and D

Understanding (lower level)

4. A summary of a recent survey is shown below:

60% liked hamburgers
70% liked pizza
40% liked both

What percentage liked neither?

* A. 10%
B. 20%
C. 30%
D. 40%

Understanding (lower level)

5. What is the probability of drawing a heart or a face card in a single random draw from a standard deck of 52 cards?

		Clubs	**Diamonds**	**Hearts**	**Spades**
Face cards	**King**	♣	♦	♥	♠
	Queen	♣	♦	♥	♠
	Jack	♣	♦	♥	♠
	10	♣	♦	♥	♠
	9	♣	♦	♥	♠
	8	♣	♦	♥	♠
	7	♣	♦	♥	♠
	6	♣	♦	♥	♠
	5	♣	♦	♥	♠
	4	♣	♦	♥	♠
	3	♣	♦	♥	♠
	2	♣	♦	♥	♠
	Ace	♣	♦	♥	♠

A. $\frac{13}{52}$

B. $\frac{12}{52}$

* C. $\frac{22}{52}$

D. $\frac{25}{52}$

Understanding (higher level)

6. Suppose you throw a pair of fair 6-sided dice. One is white and the other is black. Let T = total showing on both dice, and B = number showing on the black die. Find $P(T = 8 \mid B = 2)$.

   A. $\frac{1}{36}$

   B. $\frac{5}{36}$

   * C. $\frac{1}{6}$

   D. $\frac{1}{2}$

Understanding (higher level)

7. Each of the 11 letters from the word MATHEMATICS is placed on a separate card. A card is drawn and not replaced. A second card is drawn. What is the probability that the 2 cards chosen are both vowels?

   A. $\frac{1}{20}$

   B. $\frac{1}{10}$

   * C. $\frac{6}{55}$

   D. $\frac{16}{121}$

Understanding (higher level)

8. A game begins with two cards being dealt from a standard deck of 52 cards. To win this game, the next card dealt must be the same as either of these first two cards, or fall between them. If the first two cards are a 3 and a 10, what is the probability of winning this game?

   * A. $\frac{30}{50}$

   B. $\frac{32}{50}$

   C. $\frac{30}{52}$

   D. $\frac{32}{52}$

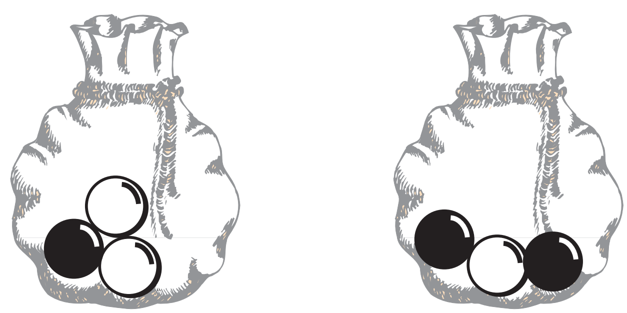
Higher Mental Processes

9. Two dart players each throw independently one dart at a target. The probability of each player hitting the bullseye is 0.3 and 0.4 respectively. What is the probability that at least one of them will hit the bullseye?

   A. 0.12
   B. 0.35
 * C. 0.58
   D. 0.70

Higher Mental Processes

10. Bag A contains 1 black and 2 white marbles, and Bag B contains 1 white and 2 black marbles. A marble is randomly chosen from Bag A and placed in Bag B. A marble is then randomly chosen from Bag B. Determine the probability that the marble selected from Bag B is white.



Bag A Bag B

   A. $\frac{5}{24}$
   B. $\frac{1}{3}$
 * C. $\frac{5}{12}$
   D. $\frac{1}{2}$

Higher Mental Processes

11. One of two cards is black on one side and white on the other side. The second card is black on both sides. One card is selected at random and the side facing up is black. What is the probability that the other side of the card is white?

    A. $\frac{1}{4}$

    * B. $\frac{1}{3}$

    C. $\frac{1}{2}$

    D. $\frac{2}{3}$

Higher Mental Processes

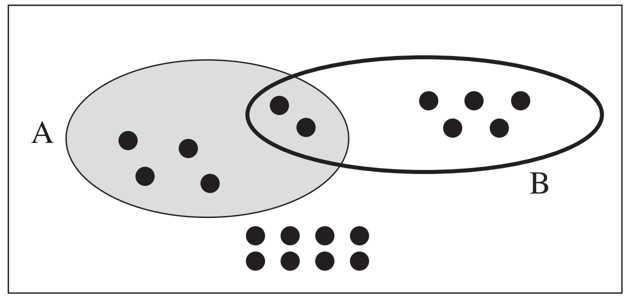
12. In a group of 30 people, what is the probability that at least 2 people will have the same birthday?

    A. 0.07
    B. 0.29
    * C. 0.71
    D. 0.93

Understanding (lower level)

13. If one of the 19 equally likely outcomes in the sample space **S** is randomly selected, find the probability that:

    a) both **A** and **B** occur.
    b) **A** but not **B** occurs.
    c) neither **A** nor **B** occurs.
    d) at least one of **A** or **B** occurs.
    e) at most one of **A** or **B** occurs.
    f) **A** occurs given that **B** has occurred.
    g) **B** occurs given that **A** has occurred.



solution

a) $P(A \text{ and } B) = \frac{2}{19}$

b) $P(A \text{ and } \overline{B}) = \frac{4}{19}$

c) $P(\overline{A} \text{ and } \overline{B}) = \frac{8}{19}$

d) $P(\text{at least one of } A \text{ or } B) = \frac{11}{19}$

e) $P(\text{at most one of } A \text{ or } B) = \frac{9}{19}$

f) $P(A \mid B) = \frac{2}{7}$

g) $P(B \mid A) = \frac{2}{6}$

Understanding (higher level)

14. Jar 1 has 3 red, 2 white and 5 black balls. Jar 2 has 4 red, 5 white and 3 black balls. A die is rolled and if a 1 or 2 comes up, a ball is selected from Jar 1; however, if a 3, 4, 5 or 6 comes up, a ball is selected from Jar 2.

    a) What is the probability of selecting a red ball?

    b) If a red ball is selected, what is the probability that it comes from Jar 1?

solution

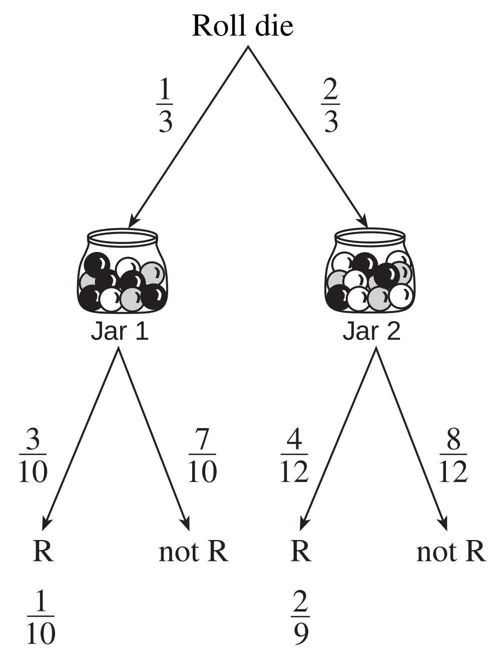
a)

Let R = red ball selected
Let J_1 = ball selected from Jar 1
Let J_2 = ball selected from Jar 2

$$\begin{aligned} P(\text{R}) &= P(\text{R and J}_1) + P(\text{R and J}_2) \\ &= P(\text{J}_1) \times P(\text{R} \mid \text{J}_1) + P(\text{J}_2) \times P(\text{R} \mid \text{J}_2) \\ &= \frac{1}{3} \times \frac{3}{10} + \frac{2}{3} \times \frac{1}{3} \\ &= \frac{1}{10} + \frac{2}{9} \\ &= \frac{29}{90} \end{aligned}$$

alternate solution

a)



$$\therefore \; P(\text{R}) = \frac{1}{10} + \frac{2}{9} = \frac{29}{90}$$

solution

b)

$$\begin{aligned} P(\text{J}_1 \mid \text{R}) &= \frac{P(\text{J}_1) \times P(\text{R} \mid \text{J}_1)}{P(\text{R})} \\ &= \frac{\frac{1}{3} \times \frac{3}{10}}{\frac{29}{90}} \\ &= \frac{9}{29} \end{aligned}$$

alternate solution

b)

Use the tree diagram above to obtain:

$$\begin{aligned} P(\text{red ball selected from Jar 1}) &= \frac{\frac{1}{10}}{\frac{1}{10} + \frac{2}{9}} \\ &= \frac{\frac{1}{10}}{\frac{29}{90}} \\ &= \frac{9}{29} \end{aligned}$$

Understanding (higher level)

15. The probability that a particular car will start on any morning is 0.9. Assuming that whether or not the car starts is independent from morning to morning, what is the probability that this car will start on at least 4 out of 5 mornings?

solution

$p = 0.9 \qquad q = 0.1 \qquad n = 5$

$$\begin{aligned} \text{P (car starts on at least 4 out of 5 mornings)} &= \text{P (car starts on 4)} + \text{P (car starts on 5)} \\ &= \text{binompdf}(5,\ 0.9,\ 4) + \text{binompdf}(5,\ 0.9,\ 5) \\ &= 0.91854 \\ &\approx 0.92 \end{aligned}$$

or

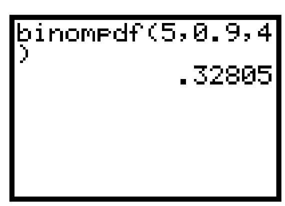
$$P(x) = {}_nC_x\, p^x q^{n-x} \quad \Rightarrow \quad P(4) \text{ or } P(5) = {}_5C_4(0.9)^4(0.1)^1 + {}_5C_5(0.9)^5(0.1)^0$$
$$= 0.91854$$
$$\approx 0.92$$

Note: The above experiment is a *binomial experiment* with 5 independent trials where the two outcomes are either: car starts (success) or car doesn't start (failure). Therefore, the binomial probability distribution can be used to solve the problem. In general, if the probability of success is p, the probability of failure is q, and the experiment is performed n times, then the probability of x successes is given by the formula: $P(x) = {}_nC_x\, p^x q^{n-x}$

Technology Note:

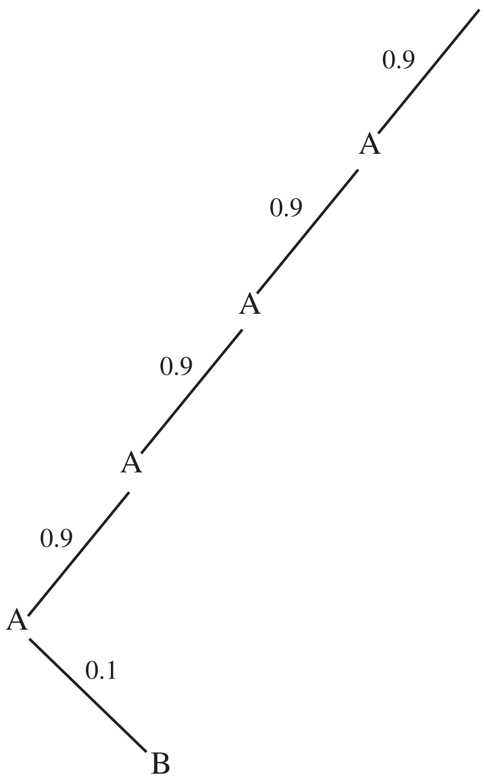
Many calculators have a built-in feature that determines the value for the binomial probability distribution. Screen images and key-strokes are provided below for reference.

Press [2nd] [DIST] [0] to obtain binompdf(, then enter n, p, x, where
n = the number of trials
p = the probability of "success"
x = number of "successes" desired



alternate solution

Let A = the car starts on any given morning, then $P(\text{A}) = 0.9$
B = the car doesn't start on any given morning $P(\text{B}) = 0.1$



To determine the probability that the car starts on at least 4 out of 5 mornings, visualize one branch of a tree diagram:

Case I: There are 5 (or $\left(\text{or } \frac{5!}{4!\,1!}\right)$) branches that contain 4 A's and 1 B.

The probability for each of these possible events is $(0.9)^4(0.1)$. Therefore, since there are 5 possible ways, the probability for these events to happen is $5(0.9)^4(0.1)$.

Case II: The car starts on exactly 5 out of 5 mornings. There is only one way this can happen: A, A, A, A, A. The probability for this event is $(0.9)^5$.

Therefore, the probability that this car starts on at least 4 out of 5 mornings is:

$$5(0.9)^4(0.1) + (0.9)^5 = 0.91854$$
$$\approx 0.92$$

Understanding (higher level)

16. A multiple-choice test has 12 questions. Each question has 4 choices, only one of which is correct. If a student answers each question by guessing randomly, find the probability that the student gets:

    a) none of the questions correct.

    b) 3 questions correct.

    c) at most 3 questions correct.

    d) at least 7 questions correct.

solution

$$p = \frac{1}{4} \qquad q = \frac{3}{4} \qquad n = 12$$

a)

$$\begin{aligned} P(\text{none of the questions correct}) &= \text{binompdf}\left(12,\ \frac{1}{4},\ 0\right) \\ &= 0.031676352 \\ &\approx 0.03 \end{aligned}$$

or

$$\begin{aligned} P(\text{none of the questions correct}) &= {}_{12}C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{12} \\ &= 0.031676352 \\ &\approx 0.03 \end{aligned}$$

b)

$$\begin{aligned} P(3\text{ questions correct}) &= \text{binompdf}\left(12,\ \frac{1}{4},\ 3\right) \\ &= 0.2581036091 \\ &\approx 0.26 \end{aligned}$$

or

$$\begin{aligned} P(3\text{ questions correct}) &= {}_{12}C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^9 \\ &= 0.2581036091 \\ &\approx 0.26 \end{aligned}$$

c) $P(\text{at most 3 questions correct}) = P(0,\ 1,\ 2,\ \text{or 3 questions correct})$

$$= \text{binomcdf}\left(12,\ \frac{1}{4},\ 3\right)$$
$$= 0.6487786174$$
$$\approx 0.65$$

or

$P(\text{at most 3 questions correct}) = P(0,\ 1,\ 2,\ \text{or 3 questions correct})$

$$= {}_{12}C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{12} + {}_{12}C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{11} + {}_{12}C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{10} + {}_{12}C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^9$$
$$= 0.6487786174$$
$$\approx 0.65$$

d) $P(\text{at least 7 questions correct}) = P(7,\ 8,\ 9,\ 10,\ 11,\ \text{or 12 questions correct})$

$$= 1 - P(0,\ 1,\ 2,\ 3,\ 4,\ \text{5 or 6 questions correct})$$
$$= 1 - \text{binomcdf}\left(12,\ \frac{1}{4},\ 6\right)$$
$$= 0.0142527818$$
$$\approx 0.01$$

or

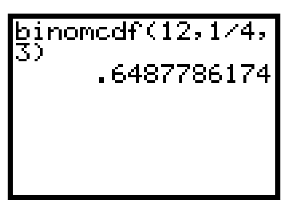
$P(\text{at least 7 questions correct}) = P(7,\ 8,\ 9,\ 10,\ \text{11 or 12 questions correct})$

$$= {}_{12}C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^5 + {}_{12}C_8\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^4 + {}_{12}C_9\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^3 + {}_{12}C_{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^2 + {}_{12}C_{11}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^1 + {}_{12}C_{12}\left(\frac{1}{4}\right)^{12}\left(\frac{3}{4}\right)^0$$
$$= 0.0142527818$$
$$\approx 0.01$$

Technology Note:

Just as many calculators have a *normal cumulative distribution function* normalcdf(, many calculators have a *binomial cumulative distribution function* binomcdf(.
Screen images and key-strokes are provided below for reference.

Press [2nd] [DIST] [A] to obtain binomcdf(, then enter $n,\ p,\ x,$ where
n = the number of trials
p = the probability of "success"
x = the maximum number of "successes" desired



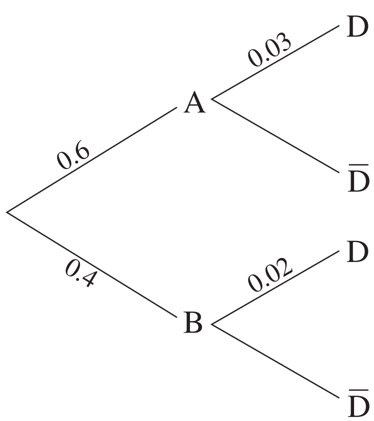
Note: Use binomcdf if the **sum** of the probabilities of obtaining $0,\ 1,\ 2,\ 3\ldots$ or x successes is desired.

Higher Mental Processes

17. Machine A produces 60% of a product while Machine B produces 40%. 3% of the production from Machine A is defective, while 2% from Machine B is defective. If a defective product is selected, what is the probability that it was produced by Machine B?

solution

Let A = product from Machine A
B = product from Machine B
D = product is defective



D → $(0.6)(0.03) = 0.018$

D → $(0.4)(0.02) = 0.008$

$$\therefore\ P(\text{defective product comes from B}) = \frac{0.008}{0.008+0.018} = 0.31$$

or

$$\frac{4}{13}$$

alternate solution

$$P(\mathrm{B}\,|\,\mathrm{D}) = \frac{P(\mathrm{B}) \times P(\mathrm{D}\,|\,\mathrm{B})}{P(\mathrm{D})}$$

$$= \frac{P(\mathrm{B}) \times P(\mathrm{D}\,|\,\mathrm{B})}{P(\mathrm{D \text{ and } A}) + P(\mathrm{D \text{ and } B})}$$

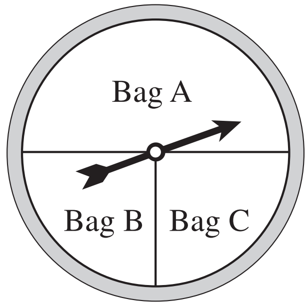
$$\frac{P(\mathrm{B}) \times P(\mathrm{D}\,|\,\mathrm{B})}{P(\mathrm{A})\,P(\mathrm{D}\,|\,\mathrm{A}) + P(\mathrm{B})\,P(\mathrm{D}\,|\,\mathrm{B})}$$

$$= \frac{0.4 \times (0.02)}{0.6(0.03) + 0.4(0.02)}$$

$$= 0.31 \quad \text{or} \quad \frac{4}{13}$$

Higher Mental Processes

18. The pointer is spun to determine a bag, and a marble is then randomly chosen from the selected bag.

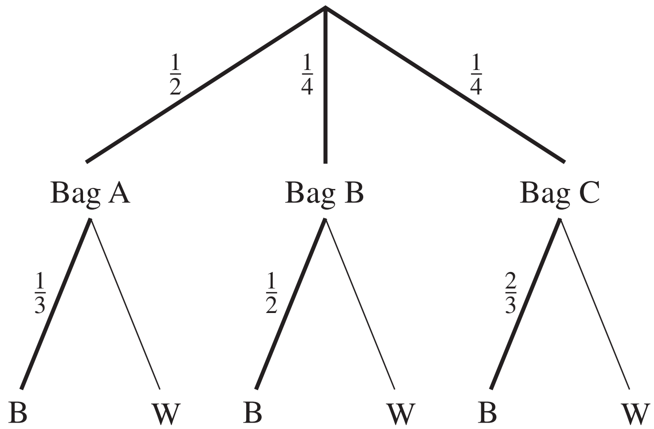


a) What is the probability that the chosen marble is black?

b) If the chosen marble is black, what is the probability that another randomly chosen marble from the same bag will also be black?

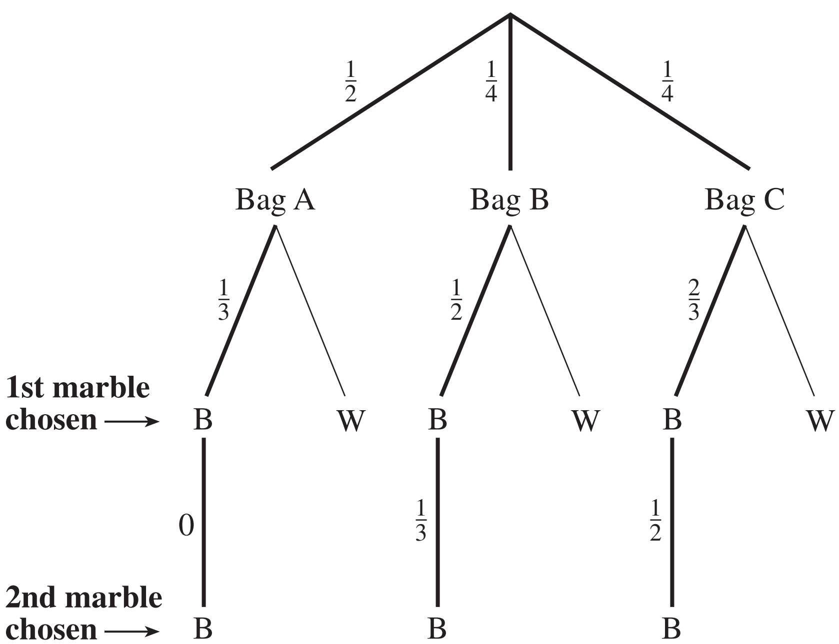
solution

a) Let B = black marble chosen   W = white marble chosen



$$P(B) = \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{4} \times \frac{2}{3}\right)$$
$$= \frac{11}{24}$$

b) Consider one more level of the tree.



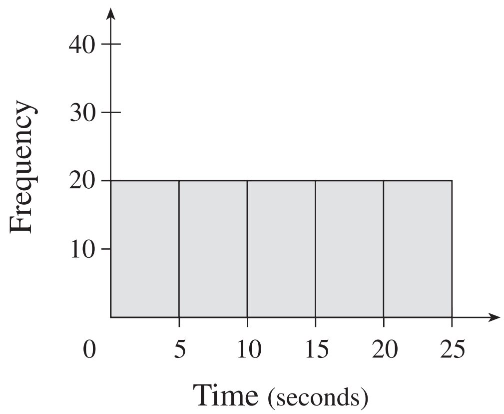
$$P(\text{second marble B} \mid \text{first marble B}) = \frac{\left(\frac{1}{2} \times \frac{1}{3} \times 0\right) + \left(\frac{1}{4} \times \frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{4} \times \frac{2}{3} \times \frac{1}{2}\right)}{\frac{11}{24}}$$
$$= \frac{3}{11}$$

G: STATISTICS AND PROBABILITY (Chance and Uncertainty) — *Statistics*

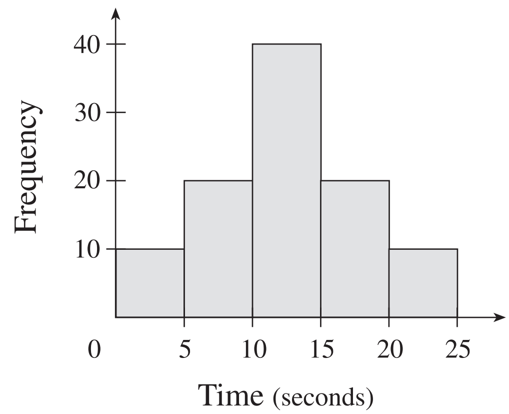
Knowledge

1. Determine which set of data would have the smallest standard deviation.

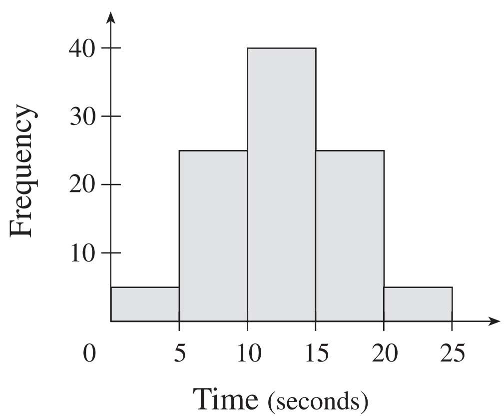
A.



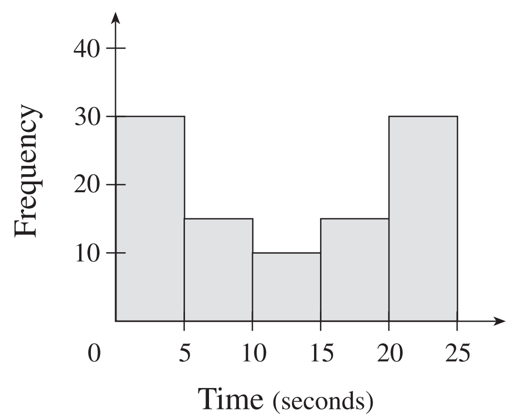
B.



* C.



D.



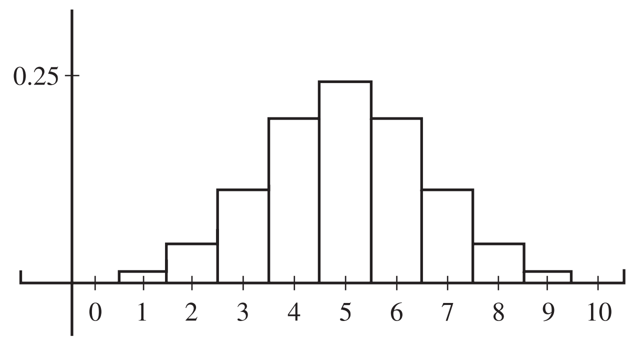
Knowledge

2. The standard normal distribution has

* A. a mean of 0 and a standard deviation of 1.
B. a mean of 0 and a standard deviation of ±1.
C. a mean of 50 and a standard deviation of 13.
D. a mean of 50 and a standard deviation of ±13.

Understanding (lower level)

3. The following graph shows the theoretical probability distribution for the number of heads obtained when a fair coin is tossed 10 times.



Determine the standard deviation of this binomial distribution.

A. 1.18
* B. 1.58
C. 2.24
D. 2.5

Understanding (lower level)

4. The area under the standard normal curve between -1.7 and 0.9 is

A. 0.768
* B. 0.771
C. 0.815
D. 0.852

Understanding (lower level)

5. The heights of students in a physical education class have a mean of 165 cm and a standard deviation of 11 cm. Calculate the z-score for a student height of 152 cm.

* A. -1.18
B. -0.85
C. 0.85
D. 1.18

Understanding (lower level)

6. Determine the value of a such that $P(Z < a) = 0.4$, where Z has a standard normal probability distribution.

A. -1.28
* B. -0.25
C. 0.25
D. 1.28

Understanding (lower level)

7. Find the standard deviation for the following population: $\{1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19\}$

* A. 5.74
B. 6.32
C. 33
D. $36.\overline{6}$

Understanding (lower level)

8. The weight of the bars of chocolate produced by a machine is normally distributed with a mean of 235 grams and a standard deviation of 4.0 grams. Determine the percentage of the chocolate bars that could be expected to weigh between 230 and 240 grams.

A. 68%
B. 78%
* C. 79%
D. 89%

Understanding (higher level)

9. The volumes of pop in cans are normally distributed with a mean of 355 mL and a standard deviation of 5 mL. Determine the range, to the nearest mL, of the volumes of the central 90% of the pop cans.

   A. between 345 and 365
 * B. between 347 and 363
   C. between 349 and 361
   D. between 350 and 360

Understanding (higher level)

10. Scores on a college entrance test are normally distributed and have a standard deviation of 36. If the probability of a score being less than 217 is 0.67, determine the mean of the scores on this college entrance test.

   A. 181
 * B. 201
   C. 233
   D. 253

Understanding (higher level)

11. Cans of oil have a mean volume of 1 000 mL and a standard deviation of 15 mL. Assuming that volumes are normally distributed, what is the probability that a randomly selected can of oil contains more than 1 020 mL?

 * A. 0.091
   B. 0.408
   C. 0.592
   D. 0.908

Understanding (higher level)

12. The results of a test are normally distributed. The mean score is 73 with a standard deviation of 8. If the top 11% of the students receive an "A", what is the minimum mark needed to receive an "A"? (Accurate to the nearest integer.)

    A. 63
  * B. 83
    C. 86
    D. 89

Higher Mental Processes

13. In a certain school district, the average grade for Communications is 63.0 and the standard deviation is 8.0. The district's grade distribution shows that 40 students with grades of 60.0 to 66.0 received a C letter grade. Assuming the grades follow a normal distribution, how many students are taking Communications in this school district?

    A. 118
    B. 135
  * C. 137
    D. 139

Higher Mental Processes

14. The marks on a grade-wide probability test had a mean of 67 and a standard deviation of 10. The marks on a grade-wide statistics test had a mean of 72 and a standard deviation of 6. If Alison obtained a mark of 82 on the probability test, what mark does she need on the statistics test to have the same z-score standing as she had on the probability test? (Accurate to the nearest integer.)

    A. 63
  * B. 81
    C. 82
    D. 84

Understanding (higher level)

15. The table below shows the number of cars per week sold by a dealership over a one-year period.

Cars sold	0	1	2	3	4
Number of weeks	2	12	17	13	8

a) Determine the mean weekly sales.

b) Determine the population standard deviation of sales.

solution

a)
$$\mu = \frac{(2)(0)+(12)(1)+(17)(2)+(13)(3)+(8)(4)}{52}$$
$$= 2.25$$

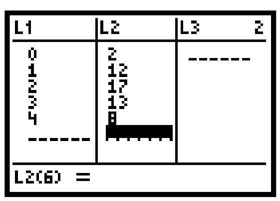
b)
$$\sigma = \sqrt{\frac{2(0-2.25)^2+12(1-2.25)^2+17(2-2.25)^2+13(3-2.25)^2+8(4-2.25)^2}{52}}$$
$$= 1.0897\ldots$$

Technology Note:

Statistical features of the graphing calculator could be used to answer this question. Screen images and key-strokes (for the TI-83 graphing calculator) are provided below for reference. (Other graphing calculators have similar capabilities, but may differ slightly in input syntax.)

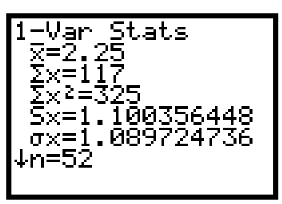
First, the number of cars sold and the number of weeks is entered into lists L_1 and L_2.

Press STAT 1



Then, use the STAT CALC menu to calculate the mean and standard deviation for lists L_1 and L_2.

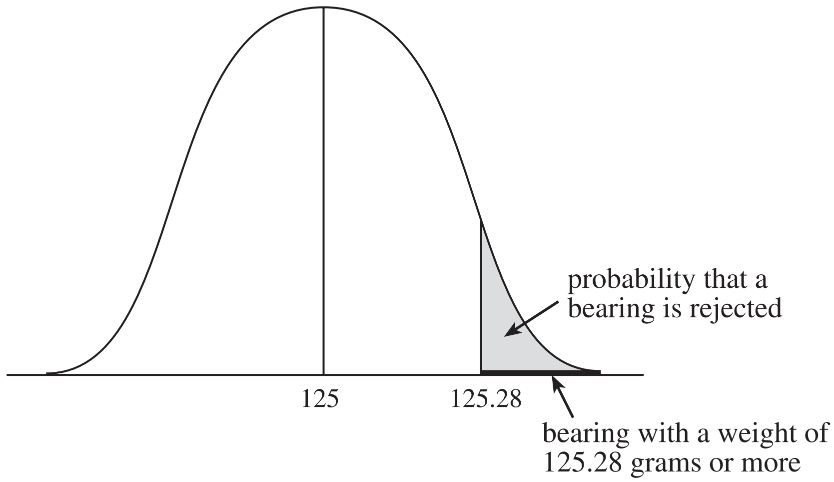
Press STAT > 1 2nd 1 , 2nd 2 ENTER



The TI-83 graphing calculator uses the symbols $\bar{x}$ for the mean and σx for the standard deviation. From the screen, $\mu = 2.25$ and $\sigma = 1.0897\ldots$

Understanding (higher level)

16. The weights of ball bearings at a manufacturing plant have a normal distribution with a mean of 125 grams and a standard deviation of 0.1 grams. Any ball bearing with a weight of 125.28 grams or more is rejected. If the manufacturing plant makes 2 000 000 ball bearings, how many would you expect to be rejected?



solution

Step 1: Probability that a bearing is rejected = normalcdf $(125.28,\ 1E99,\ 125,\ 0.1) = 0.0025551906$

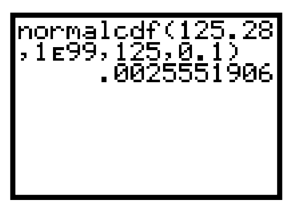
Step 2: Expected number to be rejected $= 0.0025551906 \times 2\ 000\ 000 = 5110.38$

$\therefore$ 5 110 ball bearings would be rejected

Technology Note:

Students must know how to determine the area under the standard normal curve to the right of $x = 125.28$. The screen image and key-strokes are provided below for reference.

Press [2nd] [DISTR] [2] to obtain normalcdf(, then enter *lower bound, upper bound, mean, standard deviation)*



alternate solution

$$z = \frac{125.28 - 125}{0.1} = \frac{0.28}{0.1} = 2.8$$

If z-tables are used to calculate the area under the standard normal curve to the right of $z = 2.8$, students would obtain 0.0026.
Therefore, the number to be rejected would be $0.0026 \times 2\ 000\ 000 = 5\ 200$ ball bearings.

Note: It should be noted that markers will be aware of both the table and calculator solutions and will not deduct marks for differences in accuracy.

Understanding (higher level)

17. A fair coin is tossed 100 times.

    a) Use the binomial distribution to find the probability of obtaining between 40 to 60 heads inclusive.

    b) Use the normal distribution to approximate the probability of obtaining between 40 to 60 heads inclusive.

solution

a) $P(H) = p = 0.5 \qquad n = 100$

$$\begin{aligned} P(\text{between 40 and 60 heads inclusive}) &= \text{binomcdf}(100,\ 0.5,\ 60) - \text{binomcdf}(100,\ 0.5,\ 39) \\ &= 0.9647998 \\ &\approx 0.965 \end{aligned}$$

b) $\mu = np = 100(0.5) = 50 \qquad \sigma = \sqrt{npq} = \sqrt{100(0.5)(0.5)} = 5$

$$\begin{aligned} P(\text{between 40 and 60 heads inclusive}) &\approx \text{normalcdf}(39.5,\ 60.5,\ 50,\ 5) \\ &= 0.9642712852 \\ &\approx 0.964 \end{aligned}$$

Note: The continuity correction is used when approximating the binomial with the normal. Since $np = (100)(0.5) = 50 > 5$ and $nq = (100)(0.5) = 50 > 5$, it is appropriate to use the normal approximation to the binomial.

alternate solution

b) If z-tables are used to find the normal approximation:

$$z_1 = \frac{39.5 - 50}{5} = -2.1 \qquad z_2 = \frac{60.5 - 50}{5} = 2.1$$

$$\begin{aligned} P(\text{between 40 and 60 heads inclusive}) &\approx P(-2.1 < Z < 2.1) \\ &= 0.9821 - 0.0179 \\ &= 0.9642 \\ &\approx 0.964 \end{aligned}$$

Understanding (higher level)

18. Eighty percent (80%) of Grade 12 students own a graphing calculator. If 64 students are randomly selected, determine the probability that exactly 50 students own a graphing calculator by using the following methods.

a) Use the binomial distribution to obtain this probability.

b) Use the normal approximation to the binomial to obtain an estimate of this probability.

solution

a) $p = 0.8 \qquad n = 64$

$$P(\text{exactly 50 own calculator}) = \text{binompdf}(64,\ 0.8,\ 50)$$
$$= 0.1119058942$$
$$\approx 0.112$$

or

$$P(x) = {}_nC_x\, p^x q^{n-x} \quad \Rightarrow \quad P(50) = {}_{64}C_{50}(0.8)^{50}(0.2)^{14}$$
$$= 0.1119058942$$
$$\approx 0.112$$

b) $\mu = np = 64(0.8) = 51.2 \qquad \sigma = \sqrt{npq} = \sqrt{64(0.8)(0.2)} = 3.2$

$$P(\text{exactly 50 own calculator}) \approx \text{normalcdf}(49.5,\ 50.5,\ 51.2,\ 3.2)$$
$$= 0.115799682$$
$$\approx 0.116$$

Note: Using the continuity correction, P(exactly 50 own calculator) is approximated by the area under the normal curve from 49.5 to 50.5.
Since $np = (64)(0.8) = 51.2 > 5$ and $nq = (64)(0.2) = 12.8 > 5$, it is appropriate to use the normal approximation to the binomial.

alternate solution

b) If z-tables are used to find the normal approximation:

$$z_1 = \frac{49.5 - 51.2}{3.2} = -0.53125 \approx -0.53 \qquad z_2 = \frac{50.5 - 51.2}{3.2} = -0.21875 \approx -0.22$$

$$\begin{aligned} P(\text{exactly 50 own calculator}) &\approx P(-0.53 < Z < -0.22) \\ &= 0.4129 - 0.2981 \\ &= 0.1148 \\ &\approx 0.115 \end{aligned}$$

Higher Mental Processes

19. Determine a simplified expression for the mean and standard deviation of the population a, $a+d$, $a+2d$, $a+3d$, where a and d can be any real number with $d>0$.

solution

The mean is: $\dfrac{a+a+d+a+2d+a+3d}{4}=\dfrac{4a+6d}{4}=a+1.5d$

The standard deviation is: $\sigma=\sqrt{\dfrac{\sum(x_i-\mu)^2}{n}}$

The terms:	a,	$a+d$,	$a+2d$,	$a+3d$
The deviation:	$-1.5d$,	$-0.5d$,	$0.5d$,	$1.5d$
Deviation squared:	$2.25d^2$,	$0.25d^2$,	$0.25d^2$,	$2.25d^2$

Sum of the squares of the deviations: $5d^2$

So the standard deviation is: $\sigma=\sqrt{\dfrac{5d^2}{4}}=\dfrac{d\sqrt{5}}{2}$

alternate solution

$$\sigma=\sqrt{\frac{(a-(a+1.5d))^2+(a+d-(a+1.5d))^2+(a+2d-(a+1.5d))^2+(a+3d-(a+1.5d))^2}{4}}$$

$$=\sqrt{\frac{(-1.5d)^2+(-0.5d)^2+(0.5d)^2+(1.5d)^2}{4}}$$

$$=\sqrt{\frac{2.25d^2+0.25d^2+0.25d^2+2.25d^2}{4}}$$

$$=\sqrt{\frac{5d^2}{4}}$$

$$=\frac{d\sqrt{5}}{2}$$

Note: In this question, the use of literals in the set of data requires students to work directly with the formulas for mean and standard deviation. It is important that students not assume that they will only be given numerical data, which could be input into their calculators. Students should have a good theoretical understanding of the meaning of the “central number” and the “dispersement” of the data around this number. The first solution to the question uses a table format to organize the deviation, the square of the deviation, the sum of the squares of the deviation, then determines the standard deviation using the formula:

$$\sigma=\sqrt{\frac{\sum(x_i-\mu)^2}{n}}$$

This method may be helpful for students to understand the concept of deviation and why, in the formula, the square root of the sum is necessary.

Understanding (higher level)
Higher Mental Processes (cross-topic)

20. Suppose that human birth weights for full-term babies are normally distributed with a mean of 3.4 kg and a standard deviation of 0.5 kg.

    a) What percentage of the birth weights is less than 2.5 kg?

    b) Find the birth weight exceeded by only the heaviest 2% of the babies.

    c) For two independent births, what is the probability that both babies weigh more than 3.0 kg?

    d) For 10 independent births, what is the probability that exactly 5 babies weigh less than 3.4 kg and 5 weigh more?

    e) For 10 independent births, what is the probability that 2 or more babies weigh less than 2.5 kg?

solution

a) $$\text{normalcdf}(-1\text{E}99,\ 2.5,\ 3.4,\ 0.5) = 0.0359302655$$
$$\approx 0.0359$$

Therefore the percentage of birth weights less than 2.5 kg is 3.59%.

b) $$\text{invNorm}(0.98,\ 3.4,\ 0.5) = 4.426874455$$
$$\approx 4.43 \text{ kg}$$

Therefore the heaviest 2% of the babies weigh more than 4.43 kg.

c) Let p = probability that one baby weighs more than 3.0 kg.
Then $p = \text{normalcdf}(3,\ 1\text{E}99,\ 3.4,\ 0.5) = 0.788144663$

Therefore the probability that both babies weigh more than 3.0 kg is equal to $p^2 = 0.6211720151$ or approximately 62.12%.

d) Let p = probability that one baby weighs less than 3.4 kg.
Then $p = 0.50$ and $q = 0.50$

Therefore for 10 independent births, the probability that exactly 5 babies weigh less than 3.4 kg and 5 weigh more is equal to ${}_{10}C_5\, p^5 q^5 = 0.24609375$ or approximately 24.61%.

Note: The calculator could also be used to determine this probability, namely:
$\text{binompdf}(10,\ 0.5,\ 5) = 0.24609375$

e) From part a), the probability that one baby weighs less than 2.5 kg is 0.0359302655. So, for 10 independent births, the probability that 2 or more babies weigh less than 2.5 kg is equal to $1 - \text{binomcdf}(10,\ 0.0359302655,\ 1) = 0.0479536093$ or approximately 4.80%.

A SUMMARY OF BASIC IDENTITIES AND FORMULAE

Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

Reciprocal and Quotient Identities:

$\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Addition Identities:

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\quad$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\quad$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Double-Angle Identities:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Formulae:

$t_n = ar^{n-1}$ $\quad$ $S_n = \dfrac{a(1-r^n)}{1-r}$ $\quad$ $S_n = \dfrac{a - r\ell}{1-r}$ $\quad$ $S = \dfrac{a}{1-r}$ $\quad$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Probability and Statistics:

${}_nP_r = \dfrac{n!}{(n-r)!}$ $\quad$ ${}_nC_r = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$ $\quad$ $t_{k+1} = {}_nC_k\, a^{n-k} b^k$

$P(\overline{A}) = 1 - P(A)$ $\quad$ $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

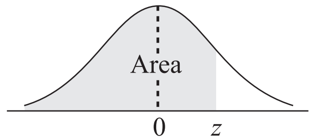
$P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$ $\quad$ $P(A \text{ and } B) = P(A) \times P(B \mid A)$

$P(x) = {}_nC_x\, p^x q^{n-x}$ $\quad$ $q = 1 - p$

$\mu = \dfrac{\sum x_i}{n}$ $\quad$ $\sigma = \sqrt{\dfrac{\sum (x_i - \mu)^2}{n}}$ $\quad$ $\mu = np$ $\quad$ $\sigma = \sqrt{npq}$ $\quad$ $z = \dfrac{x - \mu}{\sigma}$

Note: Graphing calculators will contain many of these formulae as pre-programmed functions.

THE STANDARD NORMAL DISTRIBUTION TABLE



$$F_z(z) = P[Z < z]$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

$$F_z(z) = P[Z < z]$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998