Principles of Mathematics 12

THE PROBLEM SET

September 2001

Assessment Department

ACKNOWLEDGEMENT

The Ministry of Education gratefully acknowledges the advice, assistance, and contribution of the following professionals in the development of *The Problem Set* for Principles of Mathematics 12:

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There are eight problems in *The Problem Set*. This number of problems should allow time for intense investigation of each individual problem while maintaining a variety of problem-solving experiences for the students. Effective use of these problems in a cooperative setting requires time. Teachers have found success with introducing problems at regular intervals and allowing time for students to research the ideas and consult with peers and other resource persons before discussing possible solutions.

The Problem Set is part of the **Problem Solving** organizer in Principles of Mathematics 12.

The Problem Set
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RATIONALE

Problem solving is an integral component of the Mathematics curriculum from grades 1 through 12. In Principles of Mathematics 12, the Prescribed Learning Outcome states that a student should be able to

"use a variety of methods to solve all forms of problems
(that is, real-life, practical, technical, theoretical)"

The Student Assessment and Program Evaluation Branch of the Ministry of Education develops a set of problems that reflects the intent of this learning outcome.

Through the Ministry providing a set of questions for study, both students and teachers can experience the challenge of discovering appropriate strategies that might be applied to reach a solution to a novel or unfamiliar problem.

THE NATURE OF THE PROBLEMS

It is not intended that these problems require teaching beyond the Principles of Mathematics 12 curriculum.

The problems may:

- be multi-step;
- be multi-strand;
- require a variety of problem-solving strategies;
- encourage a variety of methods of solution;
- be drawn from any topics in the Principles of Mathematics curriculum from grades 1 through 12;
- require the use of appropriate technology to assist in the problem solving process.

The problems are intended to:

- be worked, under the guidance of a teacher, by all students registered in Principles of Mathematics 12;
- provide both students and teachers with a variety of challenging problems;
- encourage students and teachers to explore mathematical problems and discover appropriate strategies;
- create an atmosphere in which students and teachers realize that, acting in concert, they can bring all their individual resources to the solution of mathematical problems;
- encourage "what if" discussions by having students and teachers modify the parameters of the original problem.

ASSESSMENT

An integral part of the study of mathematics is proficiency in solving problems and this skill should be included in the assessment of a student's progress in Principles of Mathematics 12.

NOTES TO THE TEACHER

Teachers are expected to integrate these problems into the Principles of Mathematics 12 program. Some of the problems could be introduced at the very beginning of the course, while other problems may be presented at appropriate points in the discussion of particular topics. Teachers are encouraged to pass previous problem sets to mathematics classes in other grades, where appropriate, and to use them as a source of additional problems for Principles of Mathematics 12.

Students, in general, will find the problems in this year's set challenging. It is essential that teachers play the role of facilitators with students in solving such problems. Small group work and cooperation within that group should be encouraged. Perhaps working with these problems will give students a glimpse of the excitement and ingenuity that are involved in creative thinking. Certainly students should understand that problems need not be solved in one sitting. Often it may be necessary to "play around" with a concept, put it down, and then pick it up a few days later. Students should be encouraged to try some of the problem-solving strategies to which they have been exposed in their mathematical education. It is intended that this problem-solving strand will provide students with some real-life skills and attitudes that could be transferred to their jobs in future years. As teachers, we would like our students to be thinking, reasoning human beings, rather than simply formula manipulators. Our students should be capable of relating difficult concepts and be able to acquire the confidence necessary to analyze, explore, and form conjectures when approaching any problem.

In order to act as facilitators, it is important that the teachers not become prescriptive in their approach to these problems, but allow the students leeway to solve problems in their own unique ways. It may be very desirable, though, **that teachers ask questions which may provide additional insight into the problem**. The intent is not to lead the students to a specific method of solving the problem, but to provide an entry in some cases, or to provide a context in which to consider what is being asked. Students should be encouraged to look beyond the problem at hand in order to discover a variety of techniques for arriving at a solution.

Teachers are encouraged to use a variety of strategies in the classroom in the assessment of problem solving. Some of these may include:

- having students present solutions to the class individually, in pairs, or in small groups.
- awarding marks for **complete** written solutions for problems. Teachers should reinforce the importance of style of presentation and reward students for showing multiple methods of solving problems, when possible. It is hoped that presentations will be clear, complete, and well developed. Asking for a plan or an outline of the solution path and refining the plan as work progresses may help students in solving the problem.
- having questions on examinations or quizzes which test a part of the problem or demand similar thought processes.
- asking students to keep a journal describing their progress in dealing with the problem (including ideas and attempts which were not successful).

Problem Set Questions / Corresponding Examination Questions

Previous* Problem Set question:

a) The equation of a curve is given in the following logarithmic form:

$$2 \log (x - 2) - 2 \log 4 = \log (3 - y) + \log (3 + y) - 2 \log 3$$

Write an equation of this curve in a standard form without the use of logarithms. Sketch the curve.

Corresponding examination question:

b) The graph of the function $\log(y+1) = \log(4-2x)$ must observe which pair of restrictions?

A. $y > -1,\ x > 2$
* B. $y > -1,\ x < 2$
C. $y < -1,\ x > 2$
D. $y < -1,\ x < 2$

Previous* Problem Set question:

a) What is the sum of the elements of the 100^{th} row?

$$\begin{array}{ccccccccccc} & & & & 1 & & & & & & \\ & & & 2 & + & 3 & & & & & \\ & & 4 & + & 5 & + & 6 & & & & \\ & 7 & + & 8 & + & 9 & + & 10 & & & \\ . & . & . & . & . & . & . & . & . & . & . \end{array}$$

Corresponding examination question:

b) What is the last element in the 20^{th} row for the following pattern?

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ . & . & . & . & . & . & . \end{array}$$

A. 61
B. 199
* C. 210
D. 231

***N.B.** The problem solving organizer in Principles of Mathematics 12 will be assessed on the provincial examination in two ways:

1. questions that test one part of a longer solution to a specific problem in the **September 2001 Problem Set** which the student will have studied during the year;
2. questions which draw on ideas from more than one topic area (Cross-Topic Problems).

PROBLEM 1

Continuous growth rate versus annual growth rate

> ***Definition:*** The natural logarithm of x, written $\ln x$, is defined to be the inverse function of e^x.
>
> $$\ln x = \log_e x = c \quad \Rightarrow \quad x = e^c$$

***Note*:** When giving calculator approximations, express answers accurate to 5 decimal places.

a) Use your calculator to evaluate:

i) $\ln 1$ ii) $\ln e$ iii) $\ln 10$ iv) $\ln e^3$

b) Use your calculator to evaluate:

i) e^0 ii) e^1 iii) e^{-1} iv) $e^{\ln 5}$

c) Use the results from parts a) and b) to derive a formula for $\ln e^x$ and $e^{\ln x}$.

d) Solve the following equation for k by taking the natural logarithm of both sides.

$$1.035 = e^k$$

e) A recent survey showed that the population, P, of Victoria is growing at an annual rate of 1.1%. Let P_0 represent the population on January 1, 2001 and let t represent the time, in years, since this date.

i) Express P as a function of t in the form $P = P_0 a^t$, where a is an appropriate constant.

ii) Since population can be considered to grow continuously, express the same function P as an exponential function of t using base e, in the form $P = P_0 e^{kt}$, where k is an appropriate constant.

iii) k is called the continuous growth rate. Compare this growth rate to the annual growth rate of 1.1%. Which is larger, and by how much? How significant is this difference? For example, if $P_0 = 325\,000$, determine the value of P in 10 years using each formula.

iv) The value of k found in part ii) is quite close to the annual growth rate of 1.1%. Therefore, it is often said that the model function $P = P_0 e^{rt}$ gives a good approximation for a population that is increasing annually by the rate of r%. Use this formula with $r = 1.1\%$ and $P_0 = 325\,000$ to determine the value of P in 10 years. Compare this answer to those calculated in parts i) and ii). How significant is the difference?

f) If a bank account earns interest at a rate of 6% per year compounded continuously, determine the *effective annual growth rate.*

Note: The *effective annual growth rate* is the percent increase of the initial amount over one year, assuming the amount was compounded yearly.

PROBLEM 2

Logarithms and Conics

a) Given the relation $\log(x-y)+\log(x+y)=\log 9$:

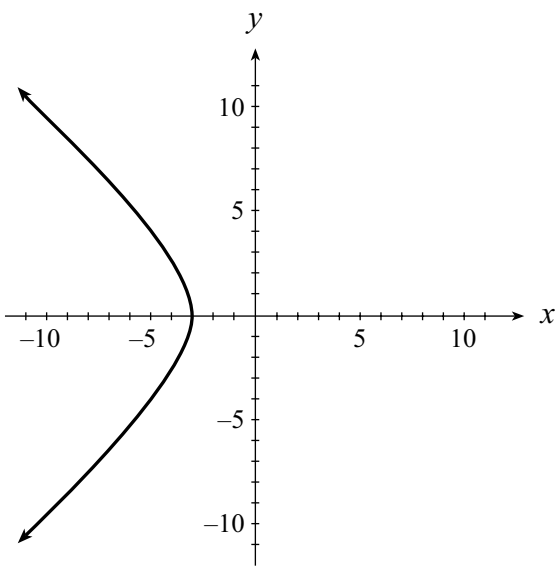
i) Determine the relation without logarithms.

ii) State the restrictions.

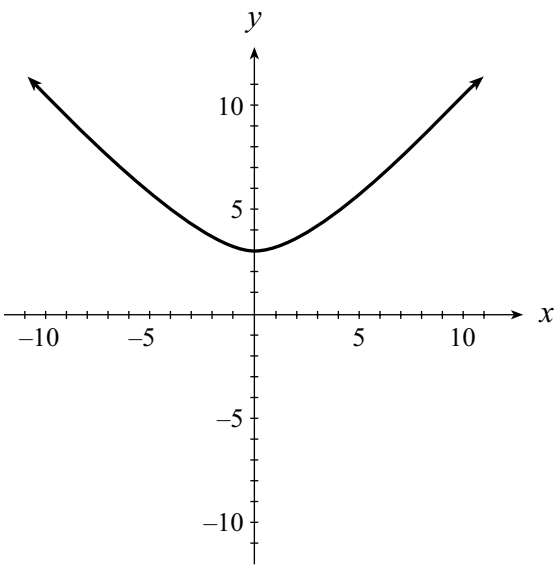
iii) Graph the relation.

b) Using logarithms, determine a relation that would produce each graph below.

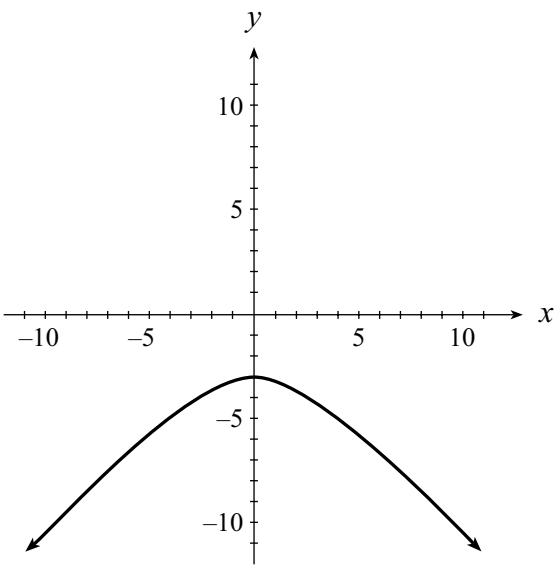
i)



ii)



iii)



PROBLEM 3

Sequences and Series

In a sequence, if $t_n = \sum_{k=1}^{n}\left(\frac{1}{x}\right)^{k-1} + \sum_{k=1}^{n}\left(-\frac{1}{x}\right)^{k-1}$:

a) Determine an expression for t_1, t_2, t_3, and t_4.

b) Determine an expression for t_{10} and the value of t_{10} if $x = 2$. (Answer in fraction form.)

c) Evaluate t_∞ if $x = 2$.

d) If $t_\infty = \frac{9}{4}$, determine the value of x.

e) If $t_\infty = p$, determine an expression for x in terms of p.